3 Species Fermion Gases

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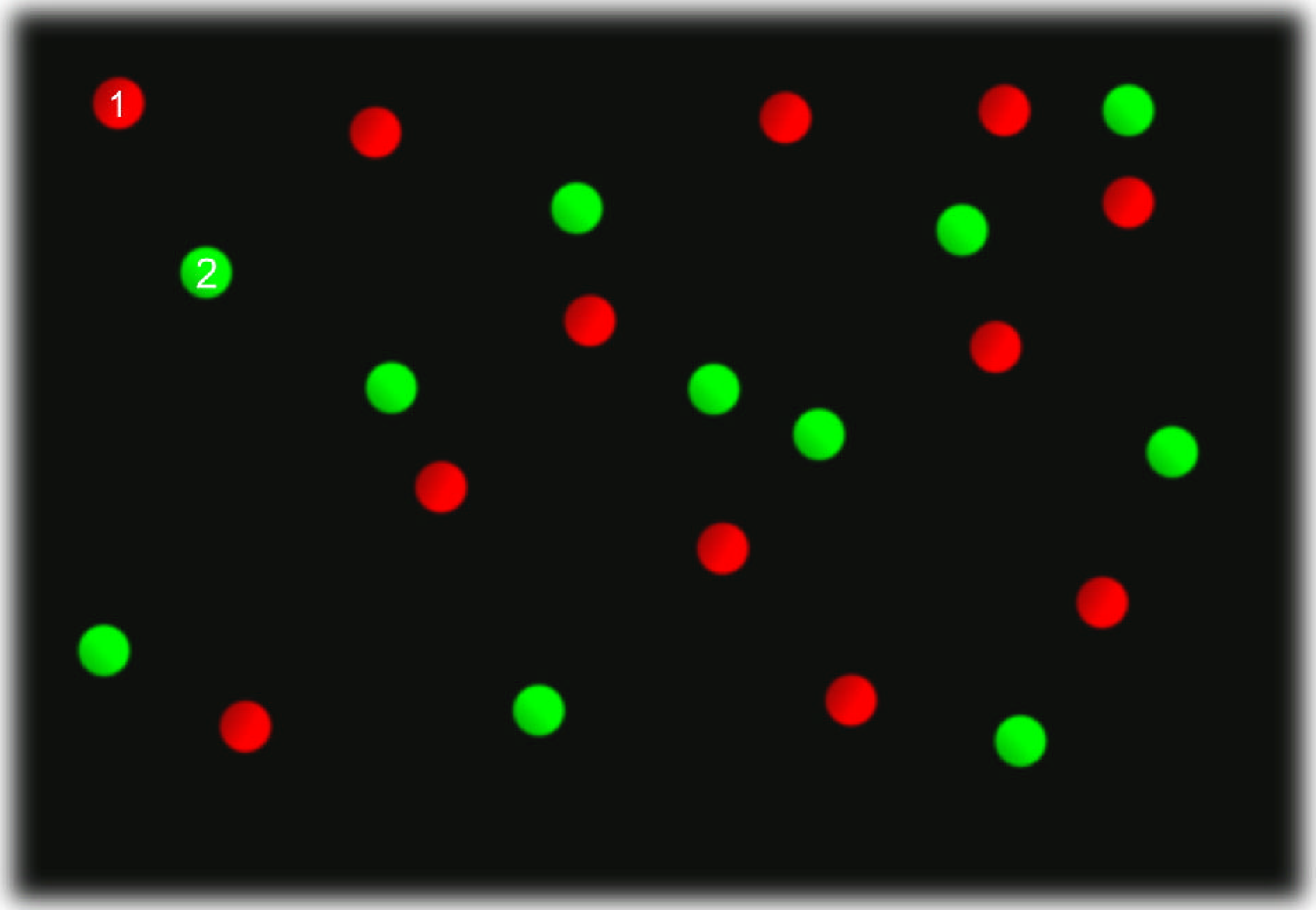
. Introduction

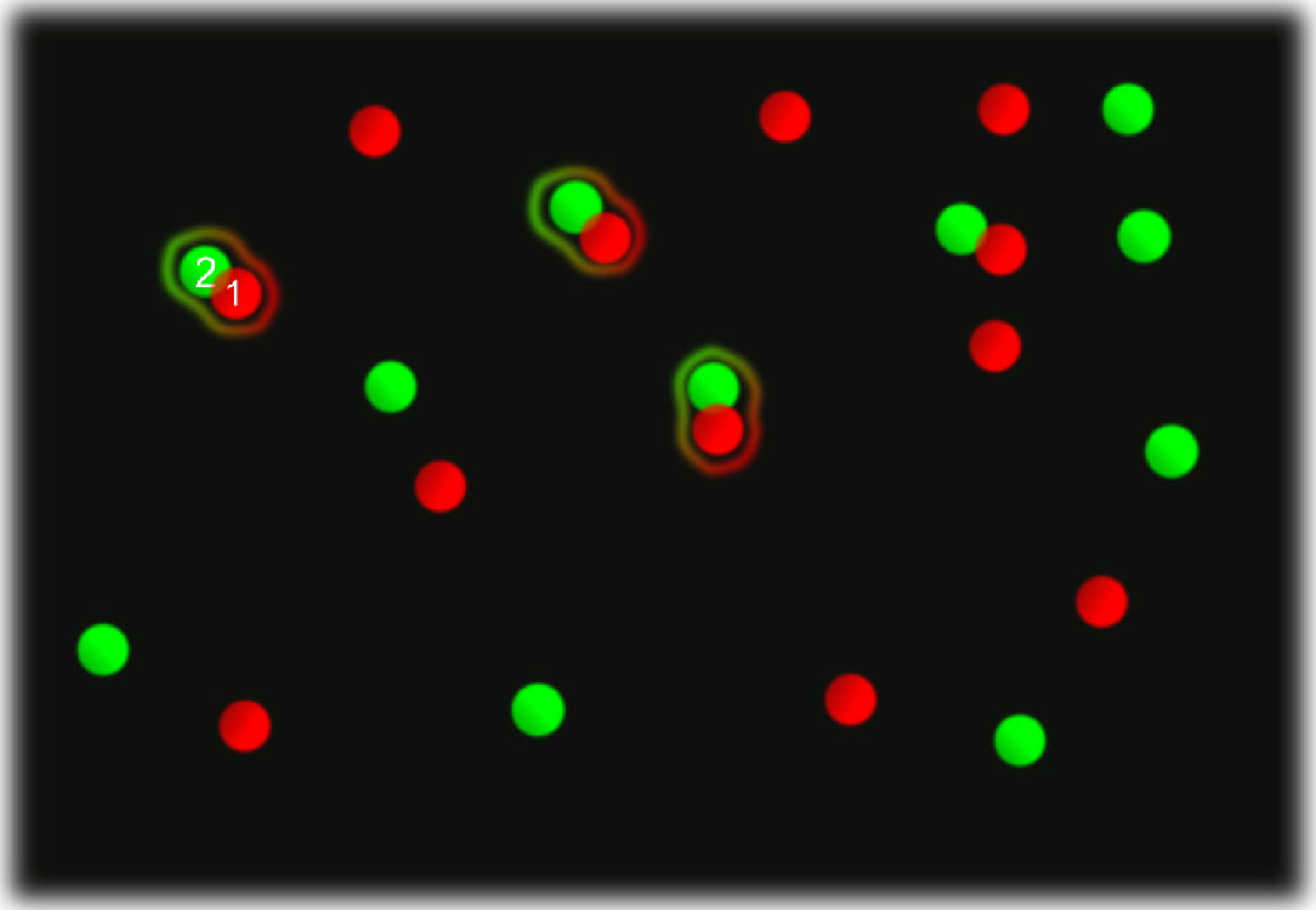
Goal:

Understanding of three fermion species systems

First: Consider a free 2 component system (no interactions)



Now add attractive interaction: Binding to bosonic states might happen



What‘s about a three fermion bound state?

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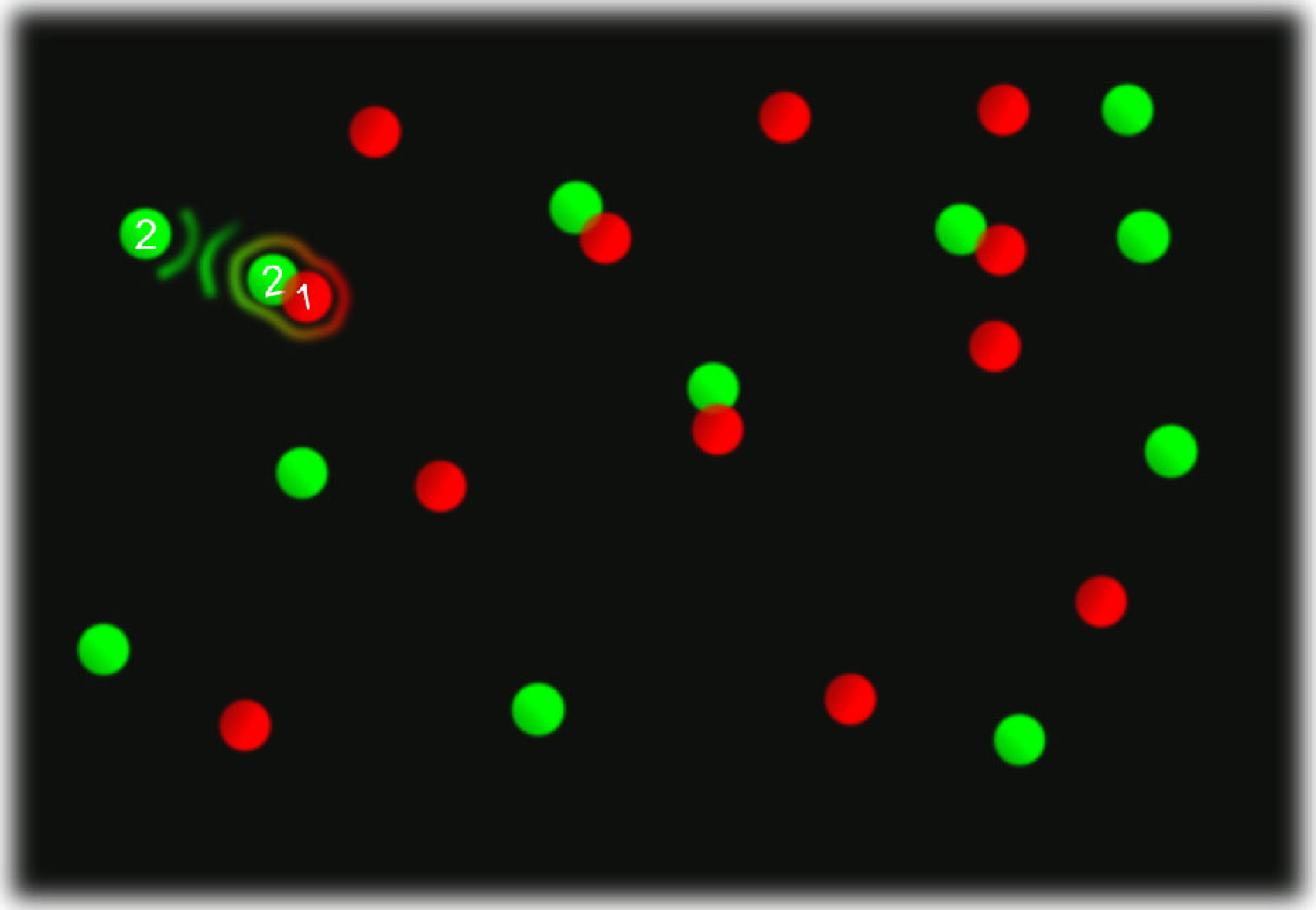
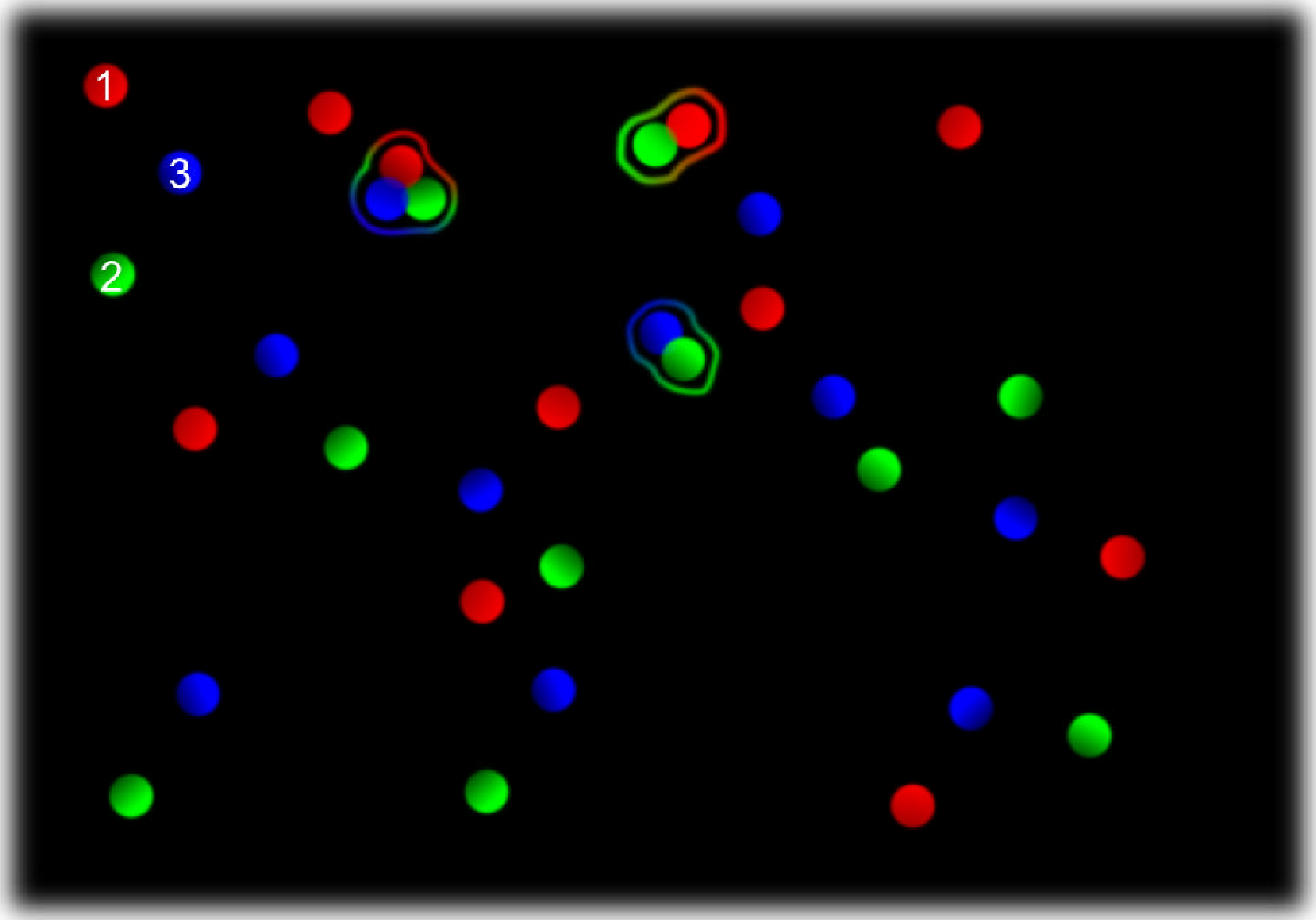


Let‘s add a third species of fermions

In principle the formation of a three fermion bound state (the *trion*) is possible.



The Question:
Do these 3 fermion bound states exist?

Remark: For bosonic systems they are shown to exist: *Efimov states*

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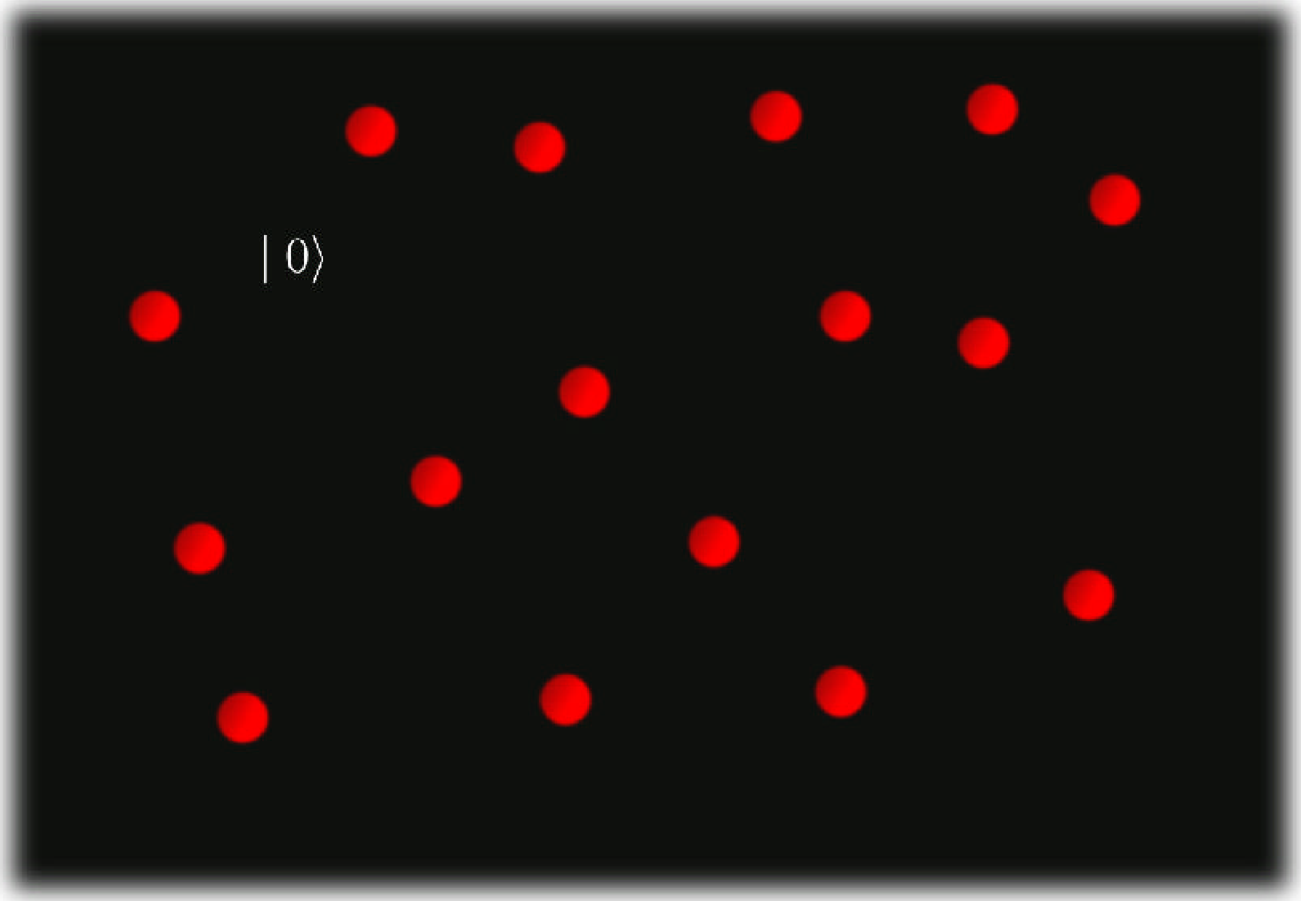
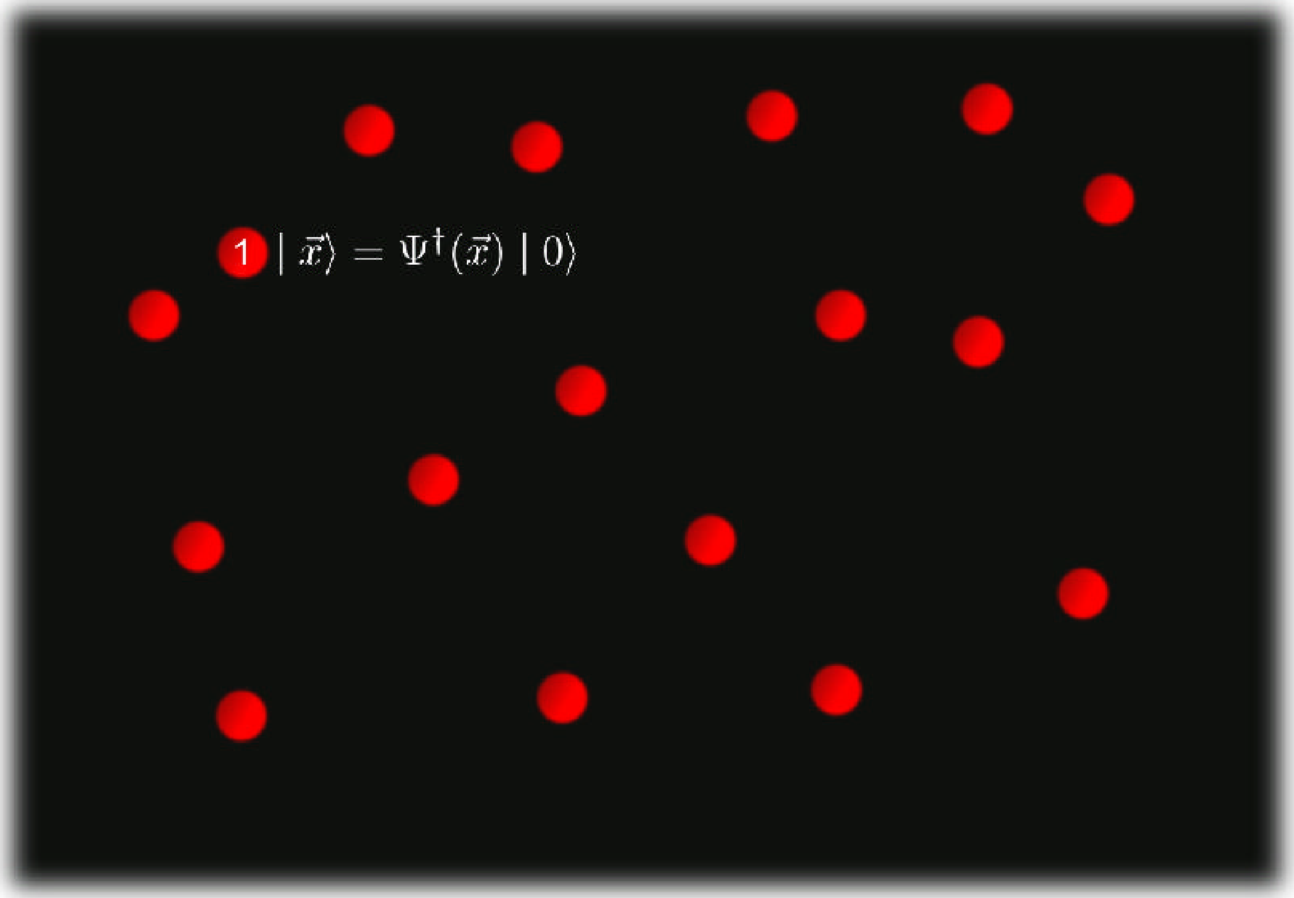
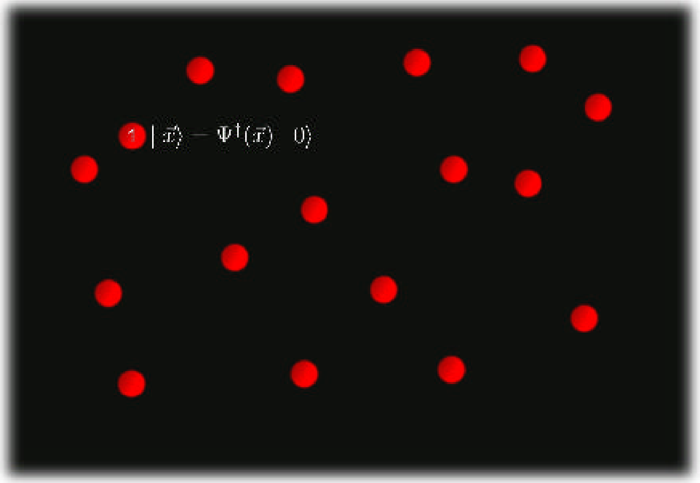
.1 Getting familiar with the Hamiltonian

.1.1 The one component gas




1 $|\,\vec{x}\rangle = \Psi^\dagger(\vec{x})\,|\,0\rangle$



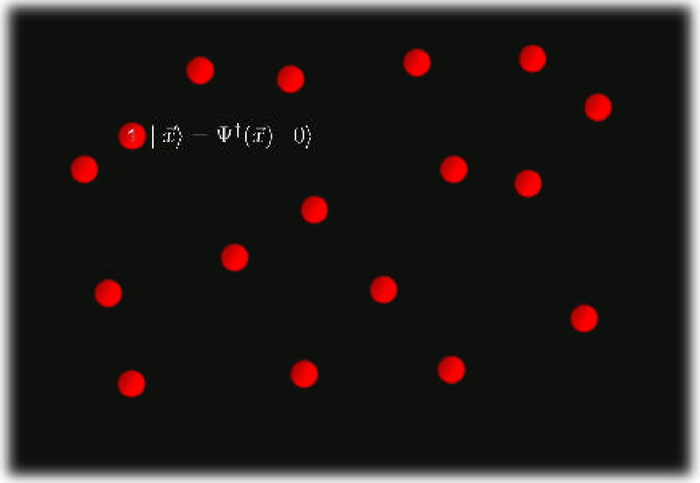
$\hat{\psi}^\dagger(\vec{x})$ applied to the vacuum creates a new particle at position x.

$\hat{\psi}(\vec{x})$ annihilates a particle at position x.

$|0\rangle$ denotes the vacuum.

$|\vec{x}\rangle$ represents a particle at position x.

How does the Hamiltonian for such a system look like?



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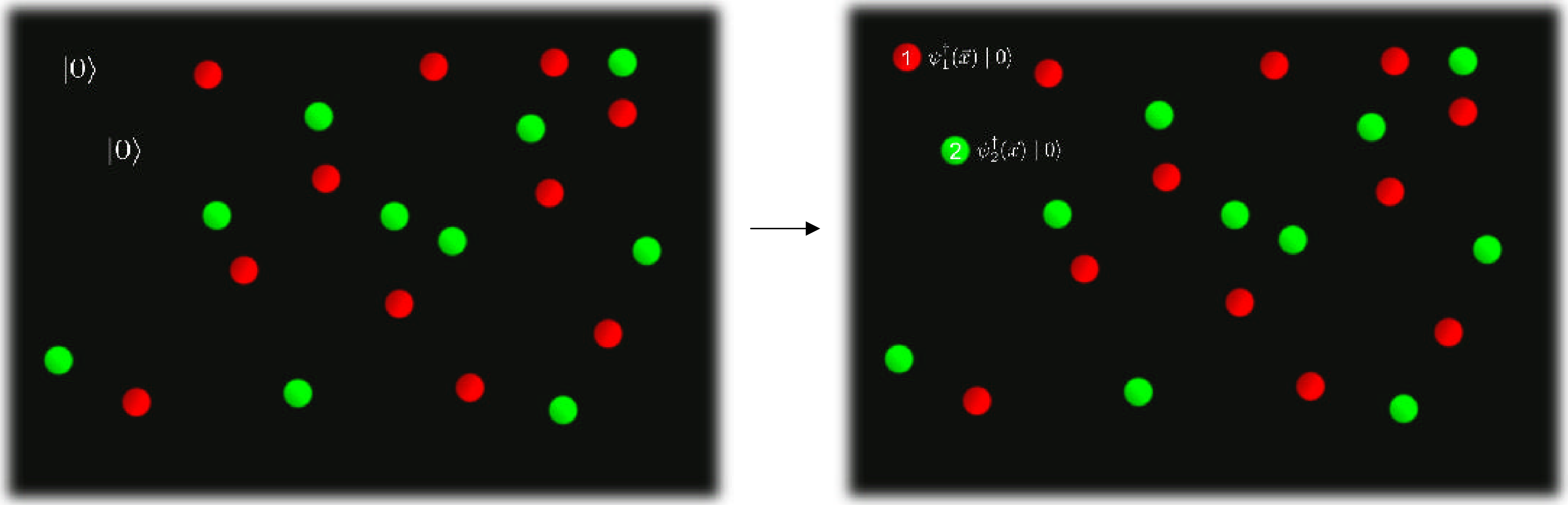
$$\hat{H} = \int d^3x \; \hat{\psi}^\dagger(\vec{x})(-\frac{\hbar^2}{2m}\Delta + V(\vec{x}))\hat{\psi}(\vec{x})$$

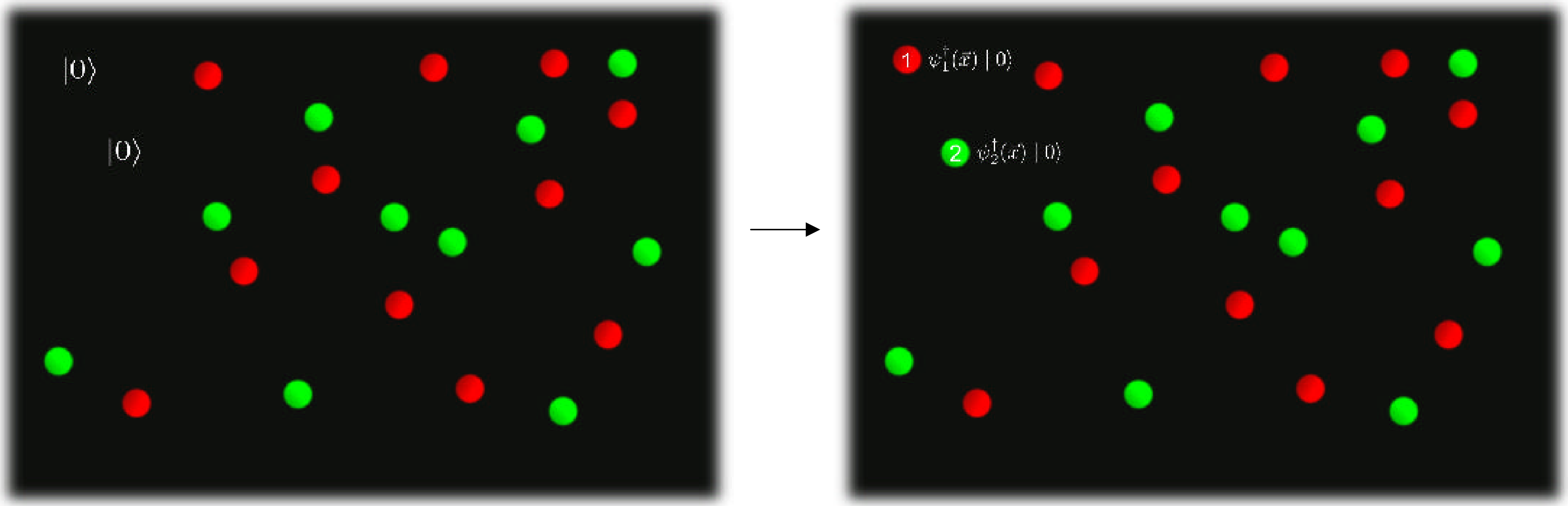
2.1.2 Two Component Gas



We need one more field in the Hamiltonian. Let`s label the fields with $\sigma = 1, 2$:

$$\hat{H} = \sum_{\sigma=1,2} \int d^3x \; \hat{\psi}_\sigma^\dagger(\vec{x})(-\frac{\hbar^2}{2m_\sigma}\Delta))\hat{\psi}_\sigma(\vec{x})$$

But now there is also the possibility of interactions between the different fermion species. How do we incorporate this?



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.1.3 Interactions in 2-component systems

Interactions between two particles are described by a scattering process.

There are different possible scattering processes between two fermions with an attractive potential:

(a) Elastic scattering

(b) Inelastic scattering (leads to molecule formation)

In diagramatic language process (a) is described by

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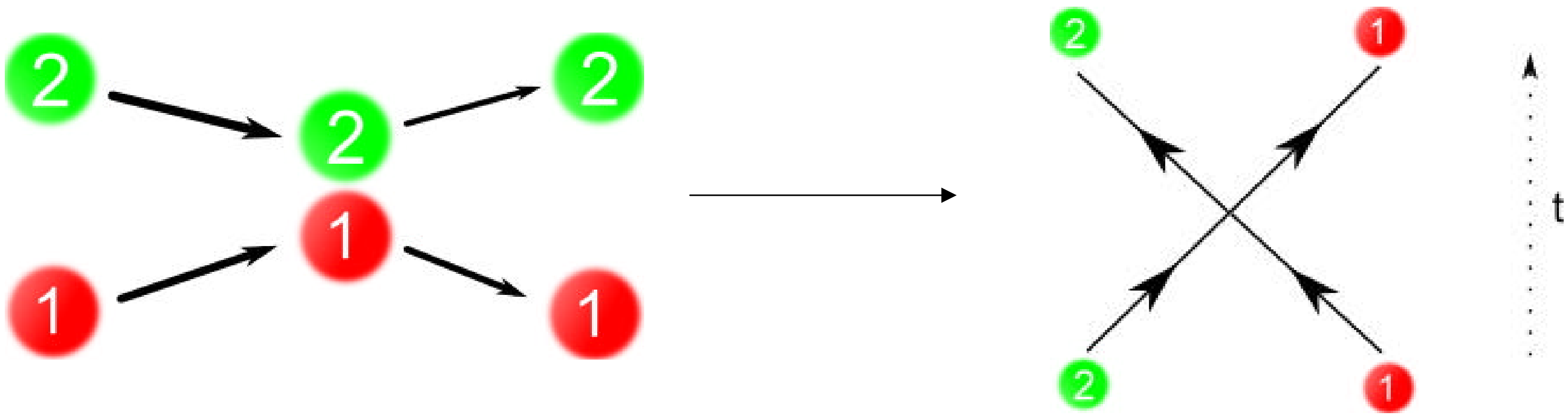

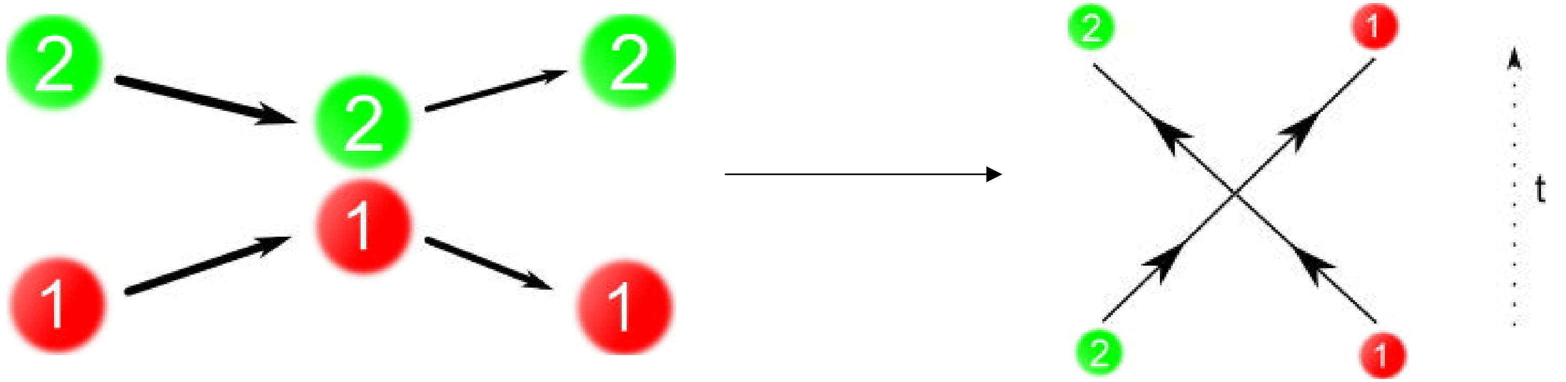

contact interaction

In the Hamiltonian this interaction corresponds to a term of the form

$$\lambda_{12}\; \hat{\psi}_1^\dagger(\vec{x})\hat{\psi}_2^\dagger(\vec{x})\hat{\psi}_2(\vec{x})\hat{\psi}_1(\vec{x})$$

$\hat{\psi}_i(\vec{x})$ describes an incoming particle.

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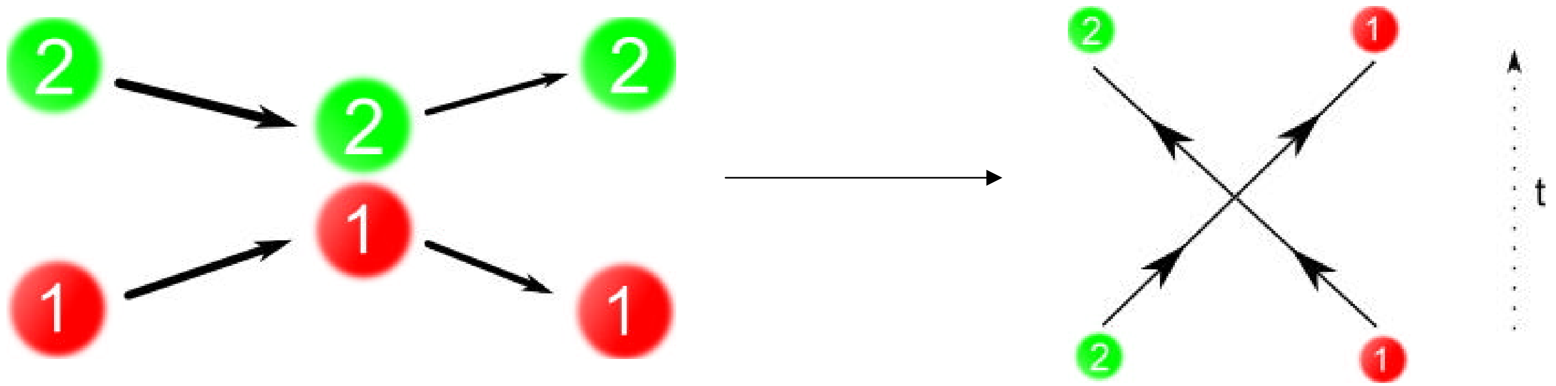
λ_{12} determines the strength of interaction ($\rightarrow$ scattering length).



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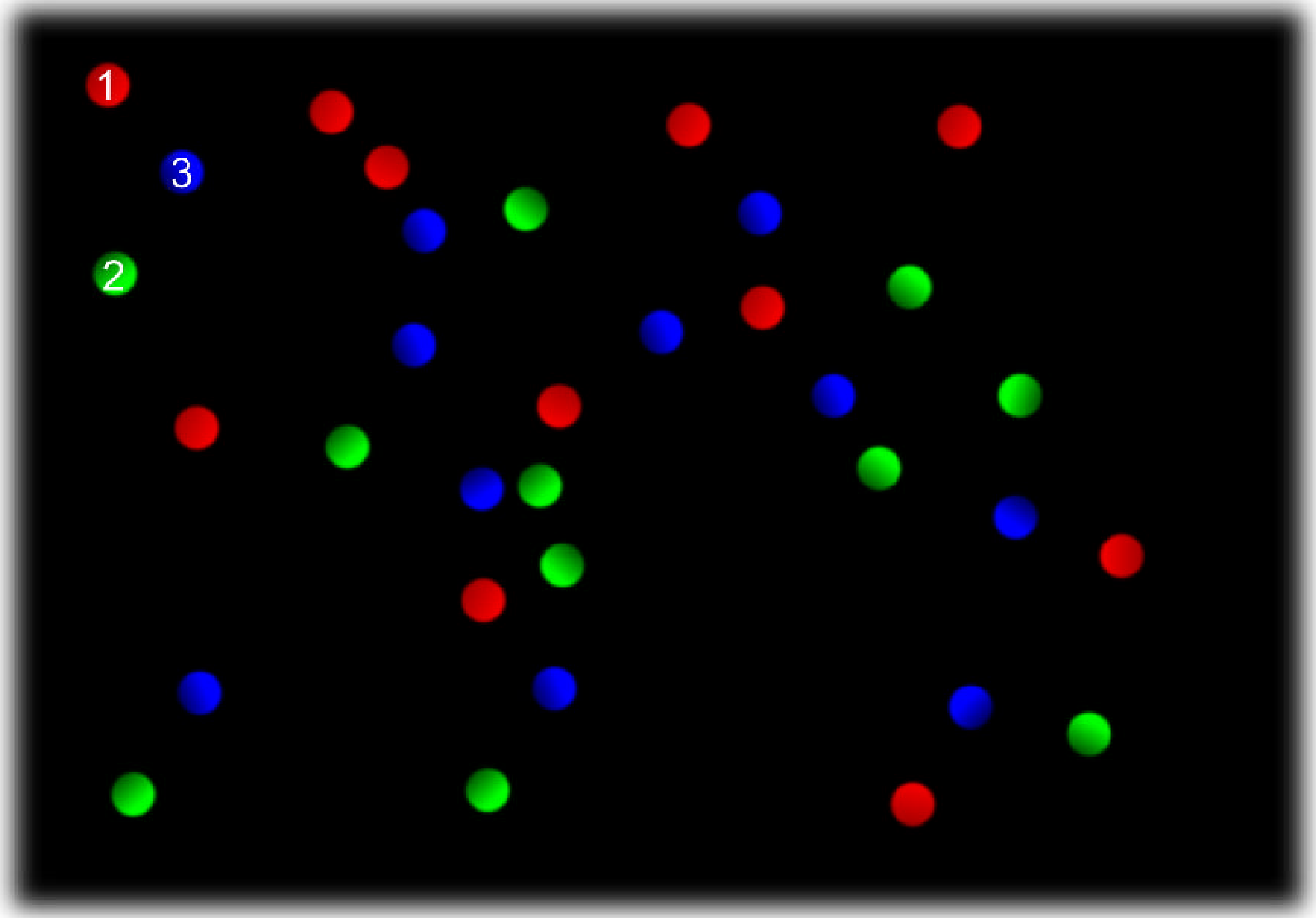
$$\rightarrow \quad \hat{H} = \sum_{\sigma=1,2} \int d^3x\ \hat{\psi}_\sigma^\dagger(\vec{x})\left(-\frac{\hbar^2}{2m_\sigma}\Delta\right)\hat{\psi}_\sigma(\vec{x}) + \int d^3x\ \lambda_{12}\ \hat{\psi}_1^\dagger(\vec{x})\hat{\psi}_2^\dagger(\vec{x})\hat{\psi}_2(\vec{x})\hat{\psi}_1(\vec{x})$$

.2 The Three Fermion system

Now we add the third fermion species



What do we need?

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& + \int d^3x\ \lambda_{12}\ \hat{\psi}_1^\dagger(\vec{x})\hat{\psi}_2^\dagger(\vec{x})\hat{\psi}_2(\vec{x})\hat{\psi}_1(\vec{x}) \\
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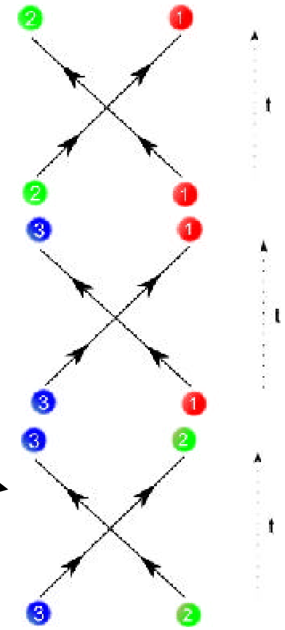
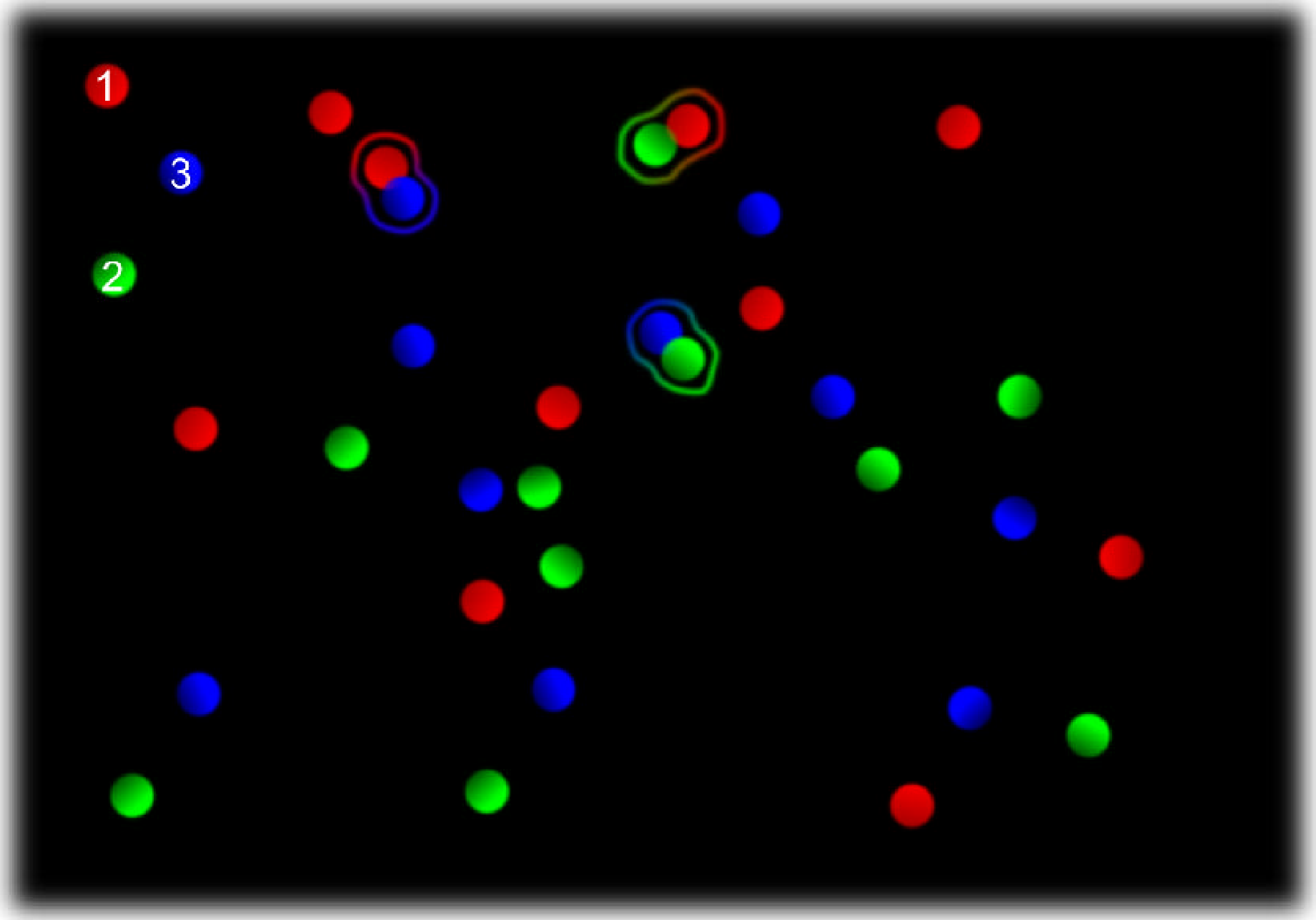


.3 A Little on BEC-BCS Theory, Statistical Field Theory, Symmetry Breaking and all that

Where are the molecules?



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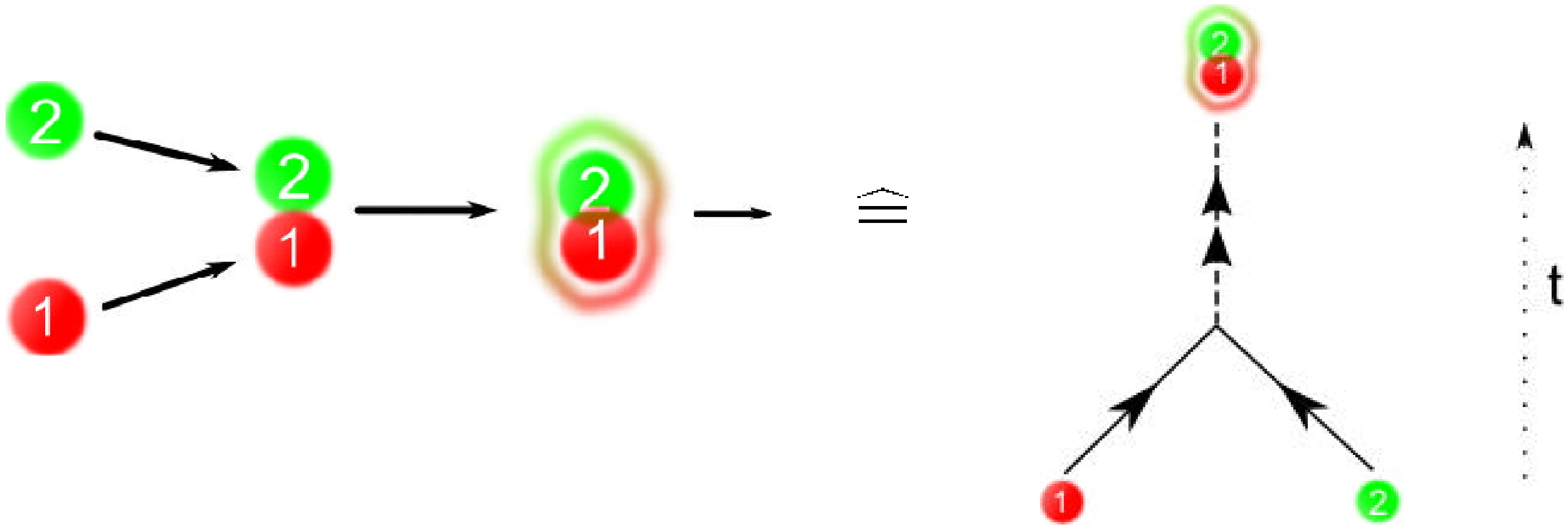

What have we done? We introduced new field(operators) $\hat{\phi}(\vec{x}), \hat{\phi}^{\dagger}(\vec{x})$ describing the creation or annihilation of a bound state of 2 fermions (molecule).

To describe the process of boson formation we need an interaction term

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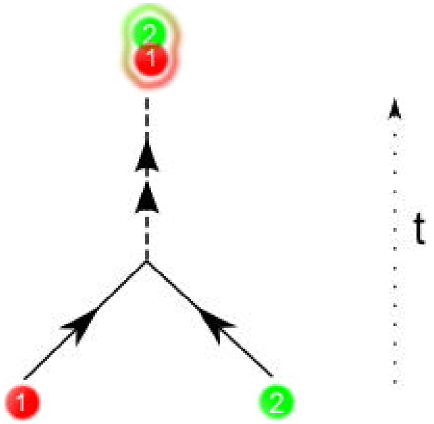

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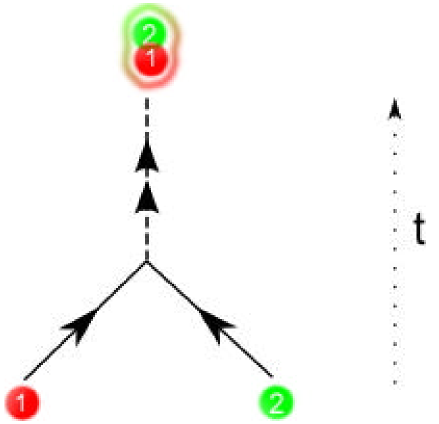

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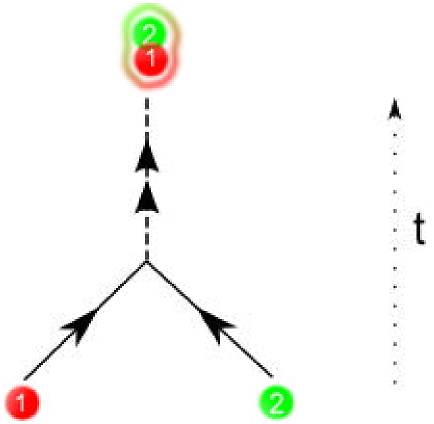

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Remark: This procedure is very much similiar to what you find in mean field BCS theory. Here you perform the replacement

$$\lambda_{12}\;\hat{\psi}_1^\dagger(\vec{x})\hat{\psi}_2^\dagger(\vec{x})\hat{\psi}_2(\vec{x})\hat{\psi}_1(\vec{x}) \rightarrow \Delta_{12}^*\hat{\psi}_2(\vec{x})\hat{\psi}_1(\vec{x})$$

$$\Delta_{12}^* = \lambda\sum_{\vec{k}}\langle\hat{\psi}^\dagger_{1,\vec{k}}\;\hat{\psi}^\dagger_{2,-\vec{k}}\rangle$$

Hold on! We missed something. There are three possibilities to form bosonic molecules:

ϕ_1

ϕ_2

ϕ_3

We have to put all these possible bosons in the Hamiltonian doing the replacements

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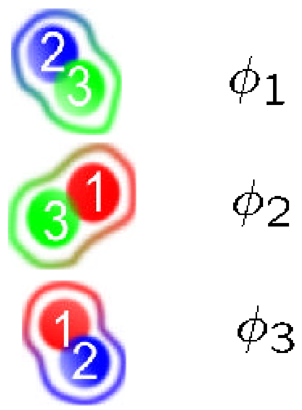

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Question: *Which bosons will develop?*

Each boson field $\widehat{\phi}_i$ can develop an expectation value, giving the corresponding particle density:

$$n_i = \langle 0|\widehat{\phi}_i^*\widehat{\phi}_i|0\rangle$$

Answer: Statistical physics – Minimize the free energy with respect to this expectation value

$$F = -k_B T \; ln(Tr[exp^{-\beta(\widehat{H}-\mu\widehat{N})}])$$

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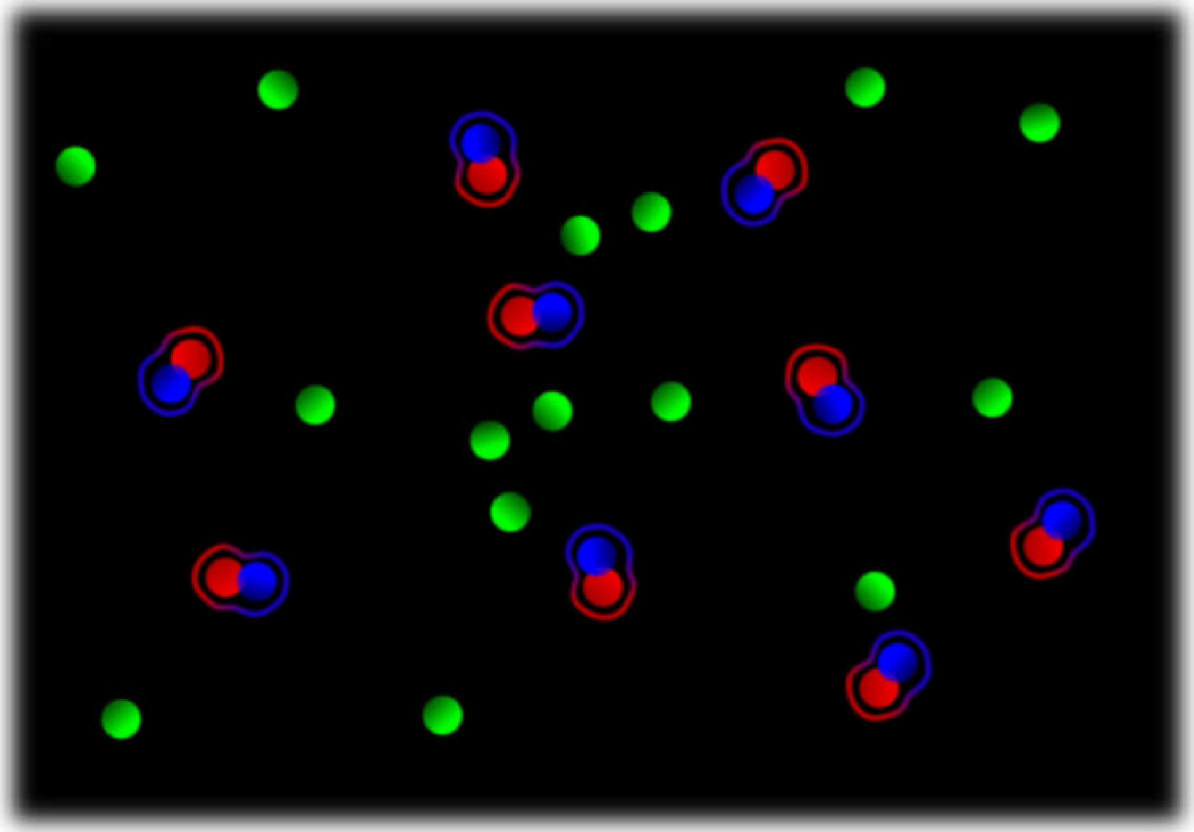

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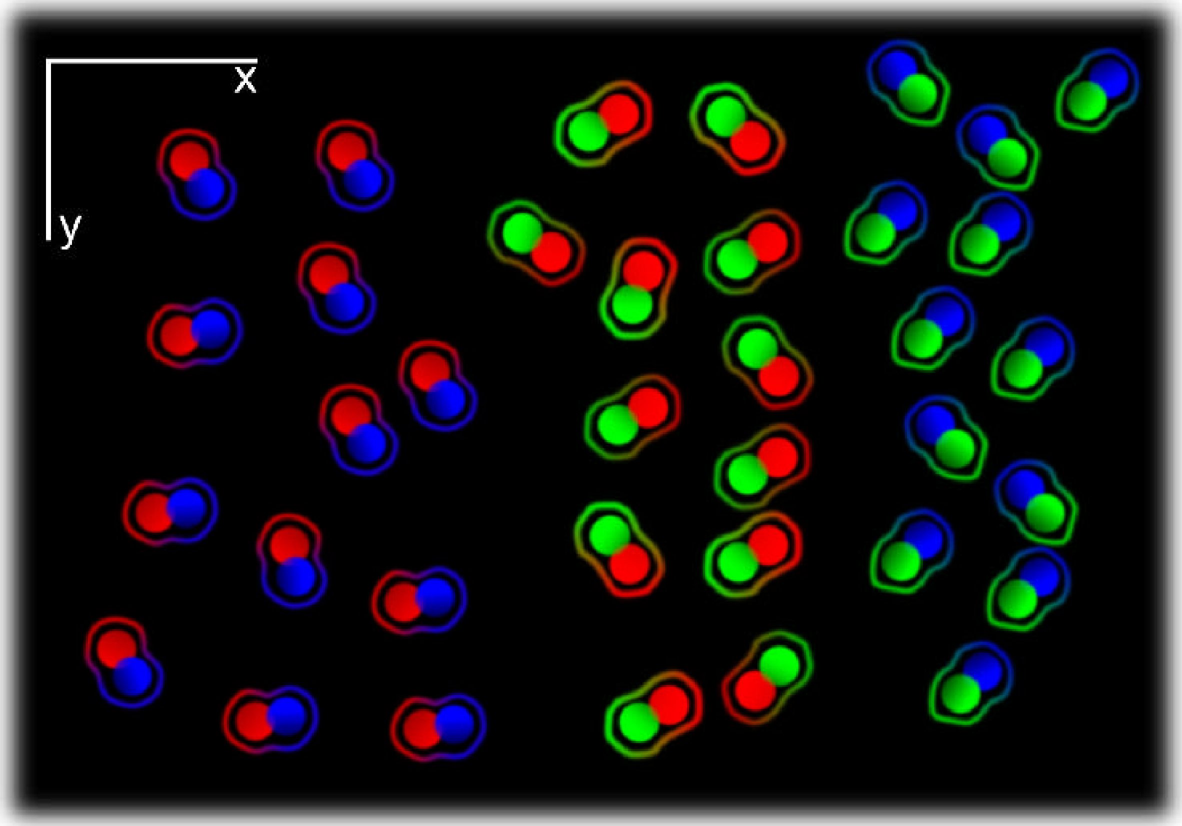

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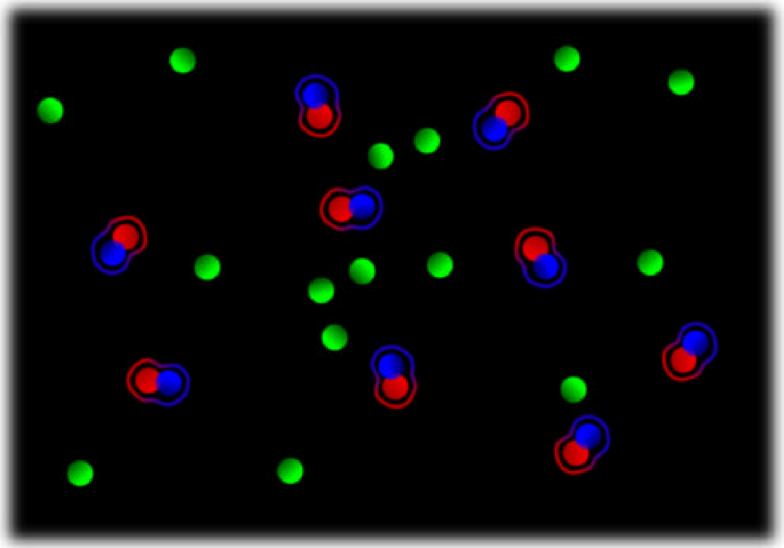
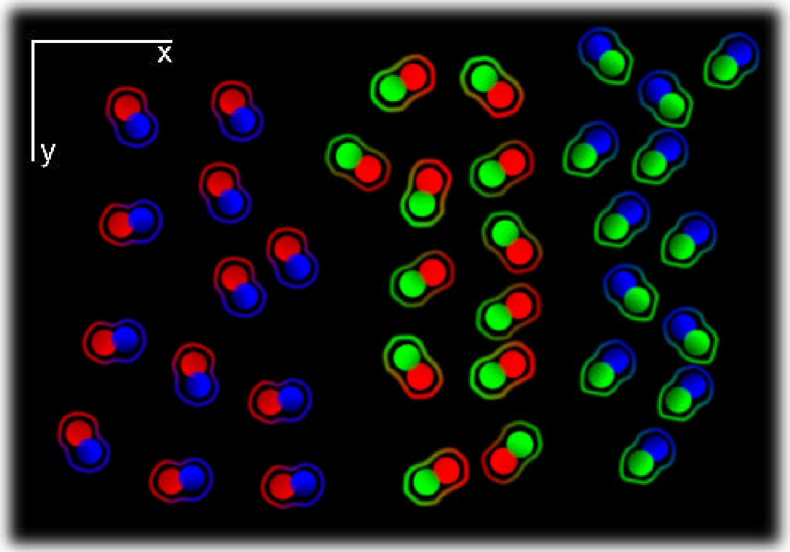




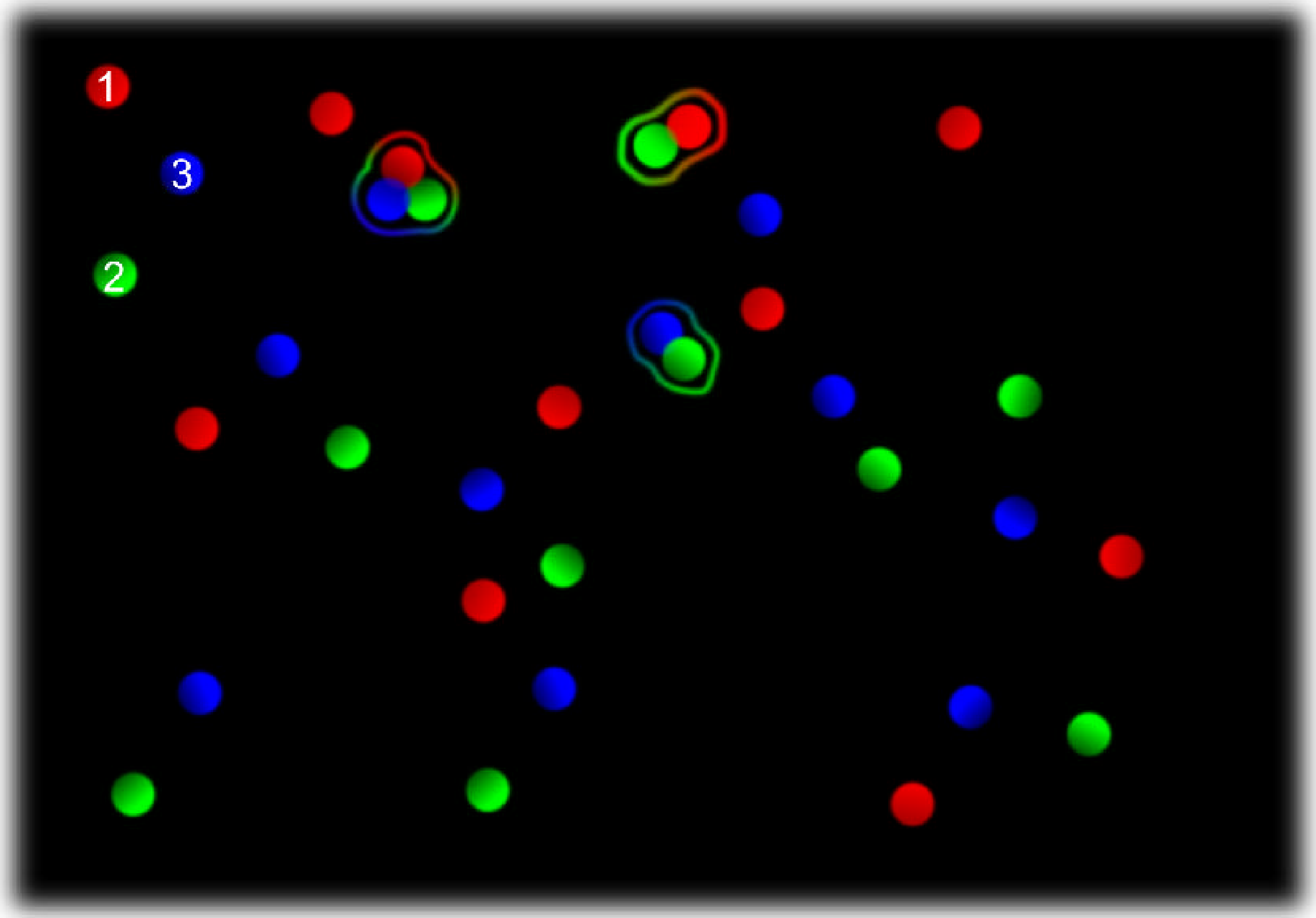
- If all bosons are energetically equivalent, nature will randomly decide for one bosonic type: *Spontaneous* symmetry breaking (SSB).

Important:

These bosons are the ones responsible for developing a BEC or BCS phase!

.4 A Glimpse on Trion Formation in Ultracold Quantum Gases – Results

What‘s about the promised *trion*?



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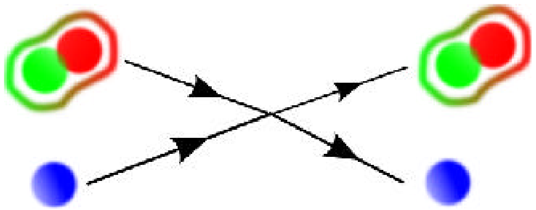
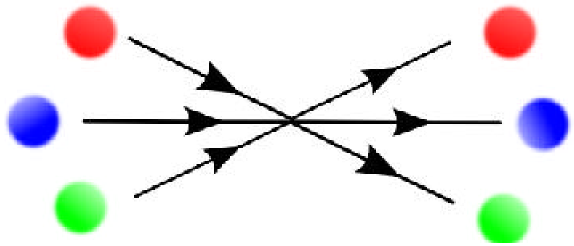


$$\propto \hat{\phi}_3^\dagger \hat{\psi}_3^\dagger \hat{\phi}_3 \hat{\psi}_3$$



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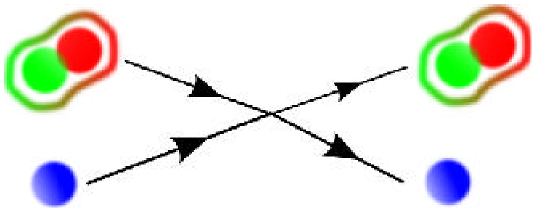

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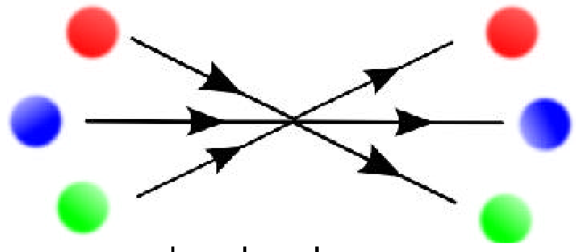

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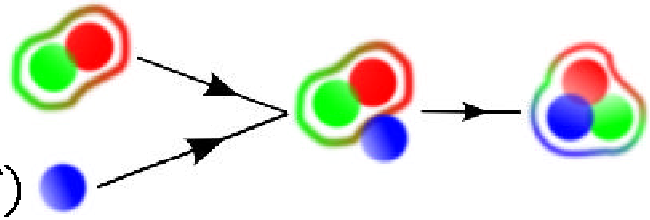
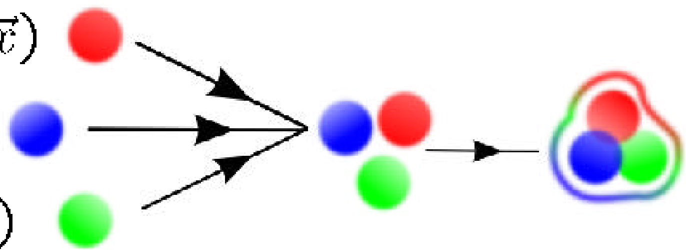


$\hat{\chi}^\dagger(\vec{x})$ Fermion!

$\hat{\psi}^\dagger(\vec{x})$, $\hat{\psi}^\dagger(\vec{x})$, $\hat{\psi}^\dagger(\vec{x})$



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The Question:

Will it be energetically favorable for the system to build these Trions?

The Hamiltonian under final consideration is

$$\hat{H} = \sum_{\sigma=1,2,3} \int d^3x\ \hat{\psi}^\dagger_\sigma(\vec{x})(-\frac{\hbar^2}{2m_\sigma}\nabla^2 + \Delta_\psi)\hat{\psi}_\sigma(\vec{x}) + \sum_{\sigma=1,2,3} \int d^3x\ \hat{\phi}^\dagger_\sigma(\vec{x})(-\frac{\hbar^2}{4m}\nabla^2 + \Delta_\phi)\hat{\phi}_\sigma(\vec{x})$$

$$+ \int d^3x\ \hat{\chi}^\dagger(\vec{x})(-\frac{\hbar^2}{6m}\nabla^2 + \Delta_\chi)\hat{\chi}(\vec{x})$$

$$+ \sum_{perm.} \int d^3x\ h_{12}\ (\hat{\phi}^\dagger_3(\vec{x})\hat{\psi}_1(\vec{x})\hat{\psi}_2(\vec{x}) + h.c.) + \sum_{\sigma=1,2,3} \int d^3x\ g_{\sigma\sigma}\ (\hat{\chi}^\dagger(\vec{x})\hat{\phi}_\sigma(\vec{x})\hat{\psi}_\sigma(\vec{x}) + h.c.)$$

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$$+\sum_{perm.}\int d^3x\ h_{12}\ (\hat{\phi}^\dagger_3(\vec{x})\hat{\psi}_1(\vec{x})\hat{\psi}_2(\vec{x})+h.c.) + \sum_{\sigma=1,2,3}\int d^3x\ g_{\sigma\sigma}\ (\hat{\chi}^\dagger(\vec{x})\hat{\phi}_\sigma(\vec{x})\hat{\psi}_\sigma(\vec{x})+h.c.)$$

$$H \xrightarrow{\text{Legendretransf.}} \mathcal{L} \xrightarrow{\text{Action}} S = \int \mathcal{L} \xrightarrow{\text{Path integral for Quantum Statistics}} \int \mathcal{D}e^{-S} \Rightarrow F$$

The Hamiltonian under final consideration is

$$\hat{H} = \sum_{\sigma=1,2,3} \int d^3x\ \hat{\psi}^\dagger_\sigma(\vec{x})(-\frac{\hbar^2}{2m_\sigma}\nabla^2 + \Delta_\psi)\hat{\psi}_\sigma(\vec{x}) + \sum_{\sigma=1,2,3} \int d^3x\ \hat{\phi}^\dagger_\sigma(\vec{x})(-\frac{\hbar^2}{4m}\nabla^2 + \Delta_\phi)\hat{\phi}_\sigma(\vec{x})$$
$$+ \int d^3x\ \hat{\chi}^\dagger(\vec{x})(-\frac{\hbar^2}{6m}\nabla^2 + \Delta_\chi)\hat{\chi}(\vec{x})$$
$$+ \sum_{perm.} \int d^3x\ h_{12}\ (\hat{\phi}^\dagger_3(\vec{x})\hat{\psi}_1(\vec{x})\hat{\psi}_2(\vec{x}) + h.c.) + \sum_{\sigma=1,2,3} \int d^3x\ g_{\sigma\sigma}\ (\hat{\chi}^\dagger(\vec{x})\hat{\phi}_\sigma(\vec{x})\hat{\psi}_\sigma(\vec{x}) + h.c.)$$

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Free energy as path integral really hard to compute. Other method:

Functional renormalization group (FRG)

(gives differential form of path integral)

Just plug action S into FRG to compute free energy F

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$$
\begin{aligned}
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& + \int d^3x \; \hat{\chi}^\dagger(\vec{x})\left(-\frac{\hbar^2}{6m}\nabla^2 + \Delta_\chi\right)\hat{\chi}(\vec{x}) \\
& + \sum_{perm.} \int d^3x \; h_{12} \; \left(\hat{\phi}^\dagger_3(\vec{x})\hat{\psi}_1(\vec{x})\hat{\psi}_2(\vec{x}) + h.c.\right) + \sum_{\sigma=1,2,3} \int d^3x \; g_{\sigma\sigma} \; \left(\hat{\chi}^\dagger(\vec{x})\hat{\phi}_\sigma(\vec{x})\hat{\psi}_\sigma(\vec{x}) + h.c.\right)
\end{aligned}
$$

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The Question:

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Very preliminary **Answer** for the vacuum:

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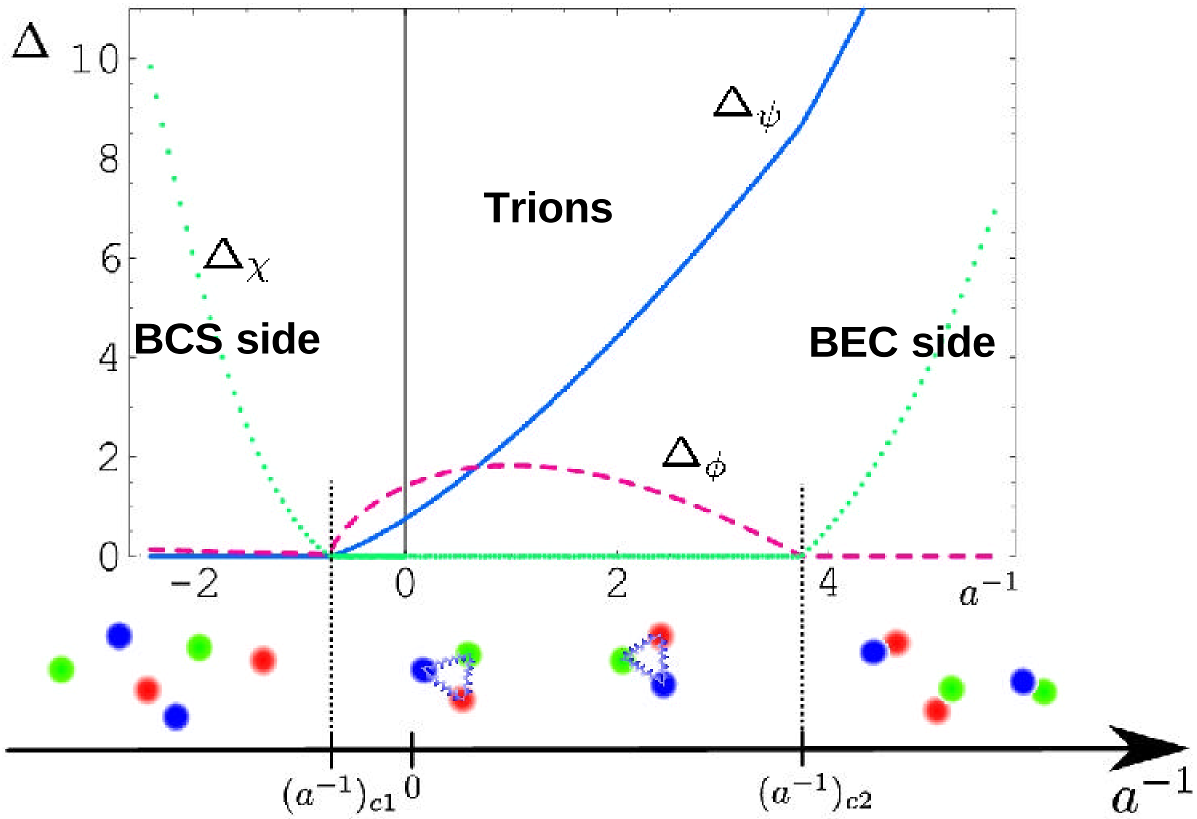
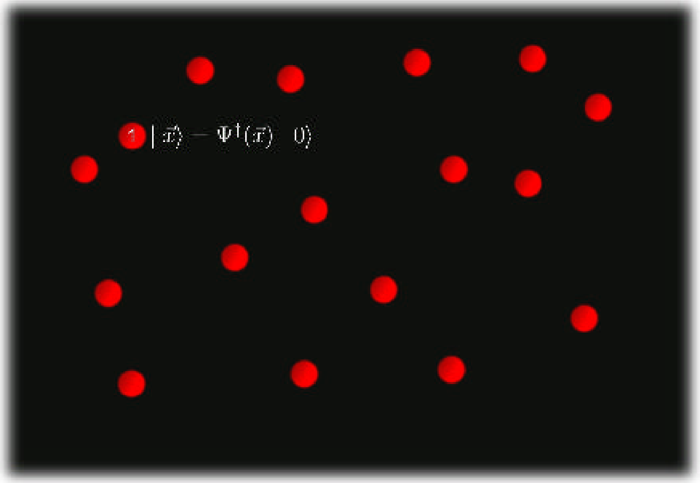


Quite possible!



$\hat{\psi}^{\dagger}(\vec{x})$ applied to the vacuum creates a new particle at position x.

$\hat{\psi}(\vec{x})$ annihilates a particle at position x.

$|0\rangle$ denotes the vacuum.

$|\vec{x}\rangle$ represents a particle at position x.

How does the Hamiltonian for such a system look like?

$$\hat{H} = \int d^3x \ \hat{\psi}^{\dagger}(\vec{x})(-\frac{\hbar^2}{2m}\Delta + V(\vec{x}))\hat{\psi}(\vec{x})$$

But we probably expected something like the Schrödinger Equation

$$i\hbar \ \partial_t \hat{\psi}(\vec{x},t) = (-\frac{\hbar^2}{2m}\Delta + V(\vec{x}))\hat{\psi}(\vec{x},t)$$

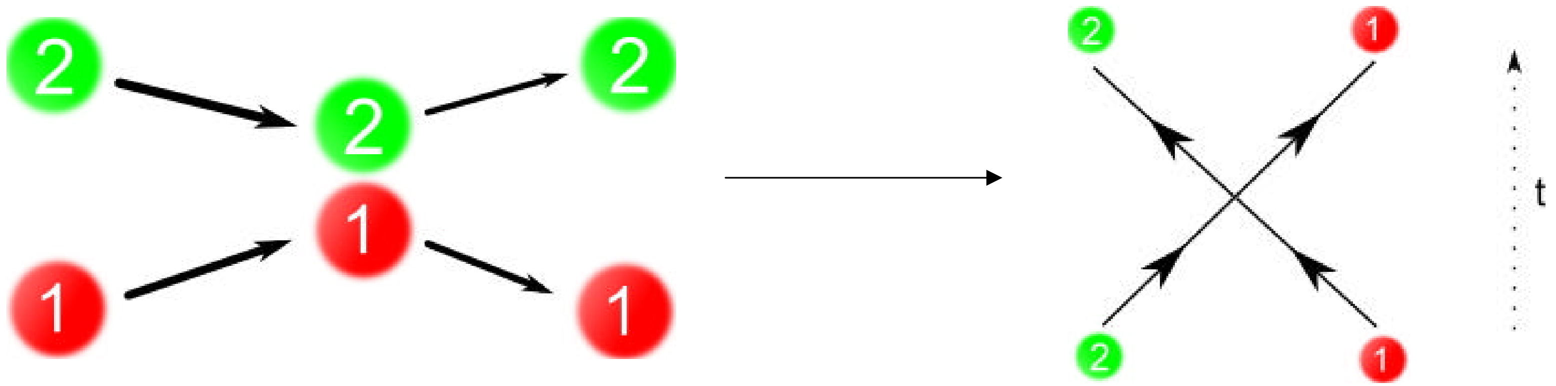
$\widehat{\psi}(\vec{x})$ is an operator. Thus it has to fulfil the Heisenberg equation of motion:

$$i\hbar\ \partial_t \hat{O} = \left[\hat{O}, \widehat{\psi}\right]$$

Plugging in $\widehat{\psi}(\vec{x})$, yields the Schrödinger Equation for the operator

$$i\hbar\ \partial_t \widehat{\psi}(\vec{x},t) = (-\frac{\hbar^2}{2m}\Delta + V(\vec{x}))\widehat{\psi}(\vec{x},t)$$

Now we should be more comfortable with this (2nd) quantization method.



In the Hamiltonian this interaction corresponds to a term of the form

$$\underbrace{\langle 0|\hat{\psi}_2(\vec{x}_2)\hat{\psi}_1(\vec{x}_1)\,[\lambda_{12}\;\hat{\psi}_1^\dagger(\vec{x})\hat{\psi}_2^\dagger(\vec{x})}_{\neq 0}\underbrace{\hat{\psi}_2(\vec{x})\hat{\psi}_1(\vec{x})]\;\;\hat{\psi}_1^\dagger(\vec{x}_1)\hat{\psi}_2^\dagger(\vec{x}_2)|0\rangle}_{\neq 0}$$

$\hat{\psi}_i(\vec{x})$ describes an incoming particle.

$\hat{\psi}_i^\dagger(\vec{x})$ describes an outgoing particle.

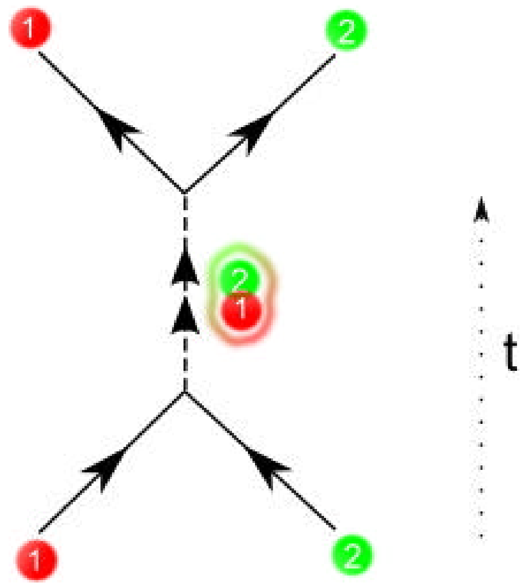
λ_{12} determines the strength of interaction ($\rightarrow$ scattering length).

Remark:

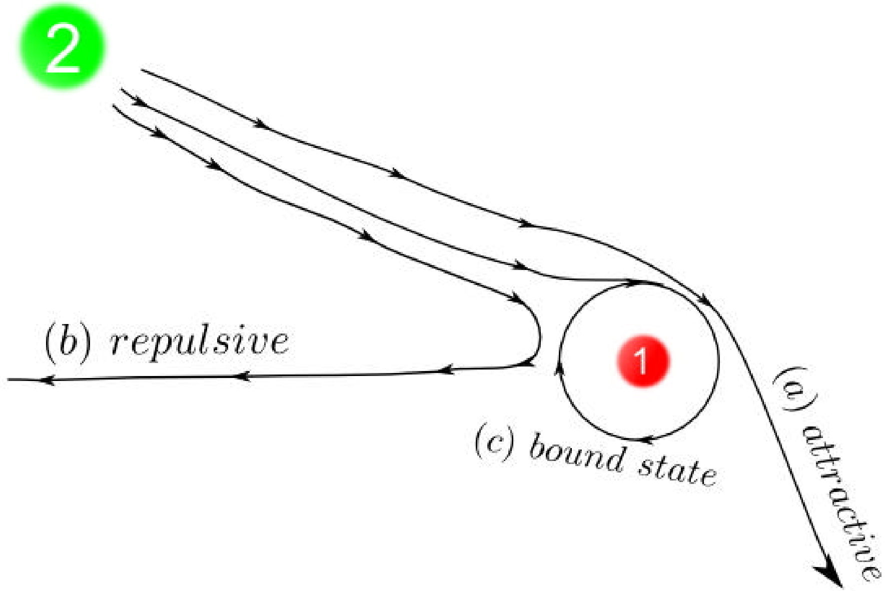
$$\begin{aligned}\hat{H} &= \sum_{\sigma=1,2,3} \int d^3x \; \hat{\psi}_\sigma^\dagger(\vec{x})\left(-\frac{\hbar^2}{2m_\sigma}\Delta\right)\hat{\psi}_\sigma(\vec{x}) \\ &+ \int d^3x \; \lambda_{12} \; \hat{\psi}_1^\dagger(\vec{x})\hat{\psi}_2^\dagger(\vec{x})\hat{\psi}_2(\vec{x})\hat{\psi}_1(\vec{x}) \\ &+ \int d^3x \; \lambda_{13} \; \hat{\psi}_1^\dagger(\vec{x})\hat{\psi}_3^\dagger(\vec{x})\hat{\psi}_3(\vec{x})\hat{\psi}_1(\vec{x}) \\ &+ \int d^3x \; \lambda_{23} \; \hat{\psi}_2^\dagger(\vec{x})\hat{\psi}_3^\dagger(\vec{x})\hat{\psi}_3(\vec{x})\hat{\psi}_2(\vec{x})\end{aligned}$$

$$\begin{aligned}\hat{H} &= \sum_{\sigma=1,2,3} \int d^3x \; \hat{\psi}_\sigma^\dagger(\vec{x})\left(-\frac{\hbar^2}{2m_\sigma}\Delta\right)\hat{\psi}_\sigma(\vec{x}) \\ &+ \int d^3x \; h_{12} \; (\hat{\phi}_3^\dagger(\vec{x})\psi_1(\vec{x})\psi_2(\vec{x}) + h.c.) \\ &- \int d^3x \; h_{13} \; (\hat{\phi}_2^\dagger(\vec{x})\psi_1(\vec{x})\psi_3(\vec{x}) + h.c.) \\ &+ \int d^3x \; h_{23} \; (\hat{\phi}_1^\dagger(\vec{x})\psi_2(\vec{x})\psi_3(\vec{x}) + h.c.)\end{aligned}$$

Where is the 4 fermion interaction? Answer:



Recall



This time we are interested in process (c), describing molecule formation.

In diagramatic language this translates to