8.5.3 Higher order corrections: Anomalous magnetic moment

1. Magnetic moment of the electron

a) Dirac equation with electron coupling to electro-magnetic field:

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu \quad \Rightarrow \quad \boxed{(i\gamma^\mu D_\mu - m)\psi = 0}$$

$$\vec{p} \to \vec{\pi} = \vec{p} - e\vec{A} \qquad \text{(canonical momentum)}$$

➡ Ansatz for the solution as for free particle:

$$\psi = \begin{pmatrix} \mathrm{X} \\ \Phi \end{pmatrix} = \begin{pmatrix} \chi\, e^{-ipx} \\ \varphi\, e^{-ipx} \end{pmatrix}$$

➡

$$i\frac{\partial}{\partial t}\mathrm{X} = \vec{\sigma}\vec{\pi}\,\Phi + (eA^0 + m)\mathrm{X}$$

$$i\frac{\partial}{\partial t}\Phi = \vec{\sigma}\vec{\pi}\,\mathrm{X} + (eA^0 - m)\Phi = 0$$

Reminder:

$$\vec{\gamma} = \gamma^0 \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

Non-relativistic limit: $E \approx m, \quad eA^0 << 2m$

For this limit it makes sense to separate interaction via charge and magnetic moment

$$i\frac{\partial}{\partial t}\chi = \vec{\sigma}\vec{\pi}\,\varphi + eA^0\chi \qquad (1)$$

$$i\frac{\partial}{\partial t}\varphi = \vec{\sigma}\vec{\pi}\,\chi + (eA^0 - 2m)\varphi \qquad (2)$$

from (2) $\varphi = \frac{\vec{\sigma}\vec{\pi}}{2m}\chi$ inserted in (1):

$$i\frac{\partial}{\partial t}\chi = \left[\frac{(\vec{\sigma}\vec{\pi})^2}{2m} + eA^0\right]\chi$$

Pauli equation.

Lower spinor component in non-relativistic limit small.

with $$(\vec{\sigma}\vec{\pi})^2 = \sigma_i \sigma_j \pi^i \pi^j = \pi^2 + \frac{1}{4}[\sigma_i, \sigma_j][\pi^i, \pi^j] = \pi^2 + e\vec{\sigma}\vec{B}$$

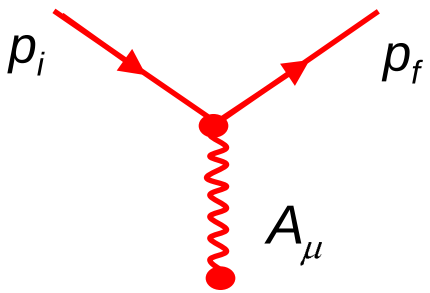
$$i\frac{\partial}{\partial t}\chi = \left[\frac{(\vec{p} - e\vec{A})^2}{2m} + \underbrace{\frac{e}{2m}\vec{\sigma}\vec{B}} + eA^0\right]\chi$$

$$= g\frac{e}{2m}\frac{\vec{\sigma}}{2}\vec{B} = g\frac{e}{2m}\vec{S}\vec{B} \quad \text{with} \quad g = 2$$

$$\langle\vec{\mu}_e\rangle = -\frac{e}{2m}\cdot g\cdot\frac{1}{2}\cdot\langle\vec{\sigma}\rangle$$

b) Gordon decomposition for electron current:

$$-e\bar{u}_f\gamma^\mu u_i \cdot A_\mu = \frac{-e}{2m}\bar{u}_f\left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]u_i \cdot A_\mu$$



Interaction of "spinless charge"

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right]$$

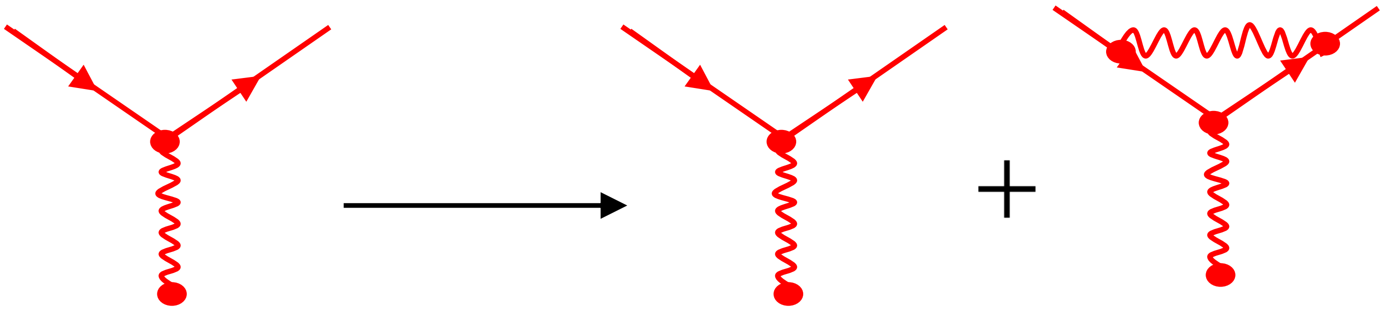
"Magnetic interaction" via spin $\rightarrow$ spin-flip

Non-relativistic limit $\chi^+\left(\frac{e}{2m}\vec{\sigma}\vec{B}\right)\chi$ since. $u = \begin{pmatrix}\chi \\ \not{\varphi}\end{pmatrix}$

2. Effect of higher order corrections

$$\frac{-e}{2m}\bar{u}_f\left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)^\nu\right]u_i\ A_\mu$$

$$\longrightarrow \frac{-e}{2m}\bar{u}_f\left((p_f+p_i)^\mu + (1+\frac{\alpha}{2\pi})i\sigma^{\mu\nu}(p_f-p_i)^\nu\right)u_i\ A_\mu$$



$$g=2 \qquad \longrightarrow \qquad g=2 \quad + \quad \frac{\alpha}{\pi}$$

1st order: $$\langle\vec{\mu}_e\rangle = -\frac{e}{2m}(2+\frac{\alpha}{\pi})\cdot\frac{1}{2}\cdot\langle\vec{\sigma}\rangle$$

$$g = 2+\frac{\alpha}{\pi}$$

$$a = \frac{g-2}{2} = \frac{\alpha}{2\pi}$$

Higher order corrections to g-2

Radiative corrections g-2 are calculated to the 4-loop level:

Feynman Graphs		
$O(\alpha)$		1
$O(\alpha^2)$		7
$O(\alpha^3)$	analytically	72
$O(\alpha^4)$	numerically	891
til $O(\alpha^4)$		971

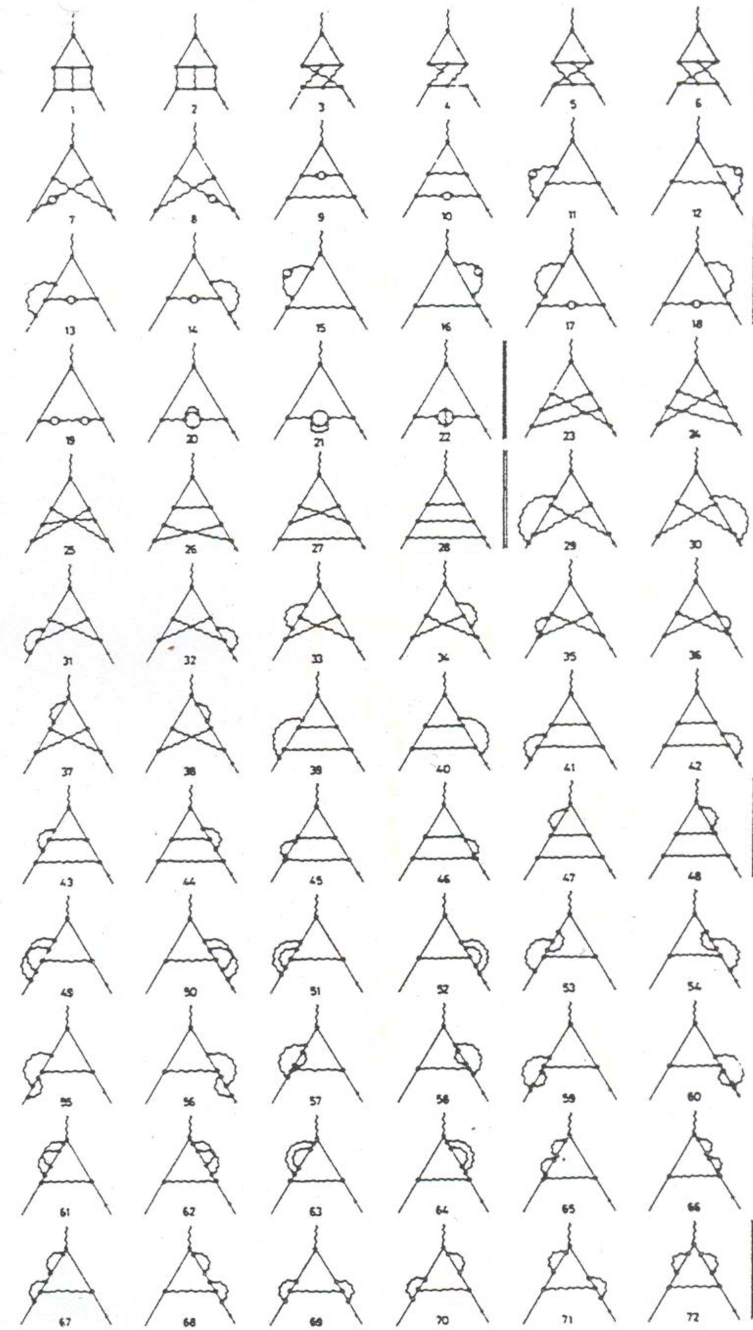


Fig. 8.2 The Feynman graphs which have to be evaluated in computing the α^3 corrections the lepton magnetic moments (after Lautrup *et al.* 1972).

Most precise QED prediction.

T. Kinoshita et al.

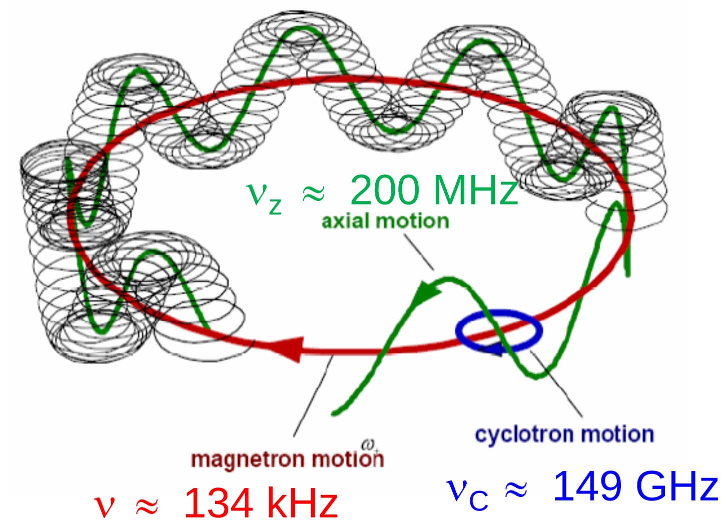
$$a = \frac{g-2}{2}$$

Kinoshita 2007

$$a_e = \frac{\alpha}{2\pi} - 0.328\ldots\left(\frac{\alpha}{\pi}\right)^2 + 1.182\ldots.\left(\frac{\alpha}{\pi}\right)^3 - 1.9144\ldots.\left(\frac{\alpha}{\pi}\right)^4$$

3. Electron g-2 measurement

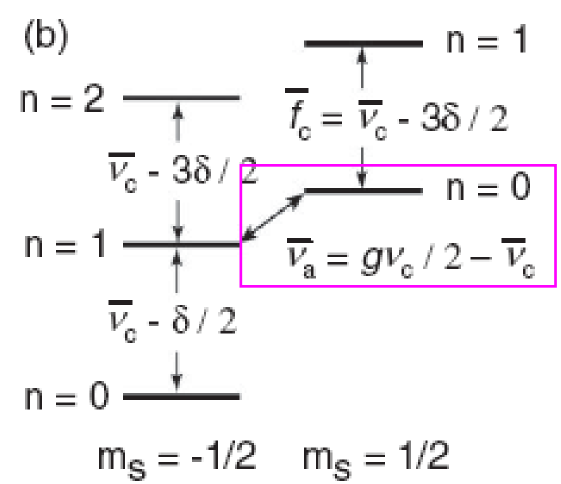
H. Dehmelt et al., 1987
G. Gabrielse et al., 2006



Experimental method:
Storage of **single** electrons in a Penning trap (electrical quadrupole + axial B field)
$\Rightarrow$ complicated electron movement (cyclotron and magnetron precessions).

Cyclotron frequency	$\omega_C = 2\frac{eB}{2mc}$
Spin precession frequency	$\omega_s = g\frac{eB}{2mc}$

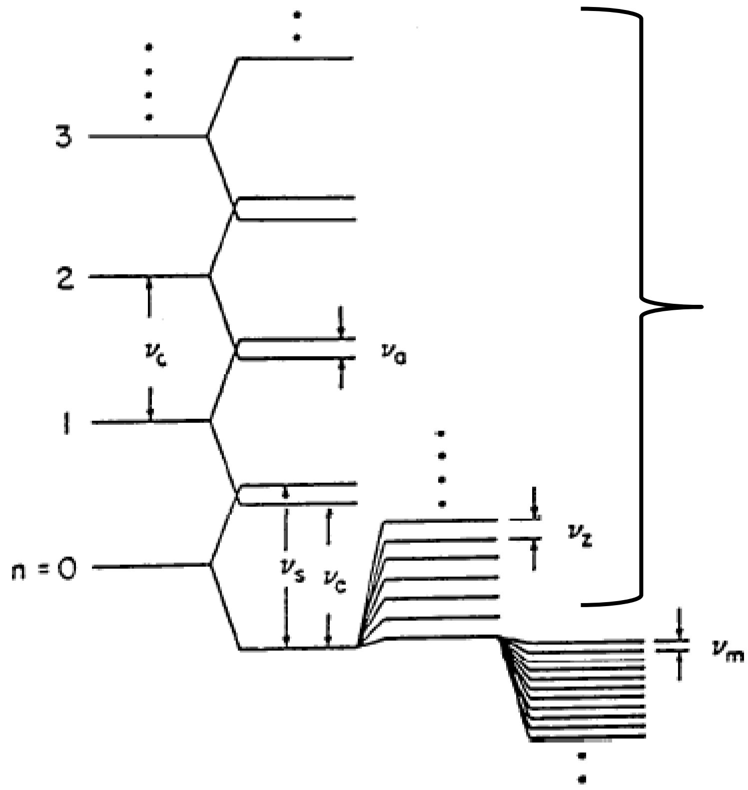
Energy levels single electron:



Idea: bound electron:

$$E(n, m_s) = \frac{g}{2}h\nu_c m_s + \left(n + \frac{1}{2}\right)h\bar{\nu}_c \underbrace{- \frac{1}{2}h\delta\left(n + \frac{1}{2} + m_s\right)^2}_{\text{Leading relativistic correction}}$$

Excitement of axial oscillation:



Cyclotron levels (n)
& Spin orientation

Axial oscillation
(E-field)

Magnetron levels
(from E-field)

http://www.nobelprize.org/nobel_prizes/physics/laureates/1989/dehmelt-lecture.pdf

Trigger RF induced transitions (ω_a) between different n states or spin flips.
(change in cyclotron or spin state revealed by axial oscillation -> feedback driven osc.)

$$\omega_a = \omega_s - \omega_c = (g-2)\mu_B B$$

$$a = \frac{g-2}{2} = \frac{\omega_s - \omega_c}{\omega_c}$$

$\Rightarrow$ most precise value of α:

$$\alpha^{-1}(a_e) = 137.035\,999\,710\,(96)$$

For comparison α from Quanten Hall

$$\alpha^{-1}(qH) = 137.036\,003\,00\,(270)$$

Phys. Rev. Lett. ***97****, 030801 (2006)*
Phys. Rev. Lett. ***97****, 030802 (2006)*

SEO = single electron oscillation

$$a_{e^-} = 0.001\,159\,652\,188\,4\,(43)$$
$$a_{e^+} = 0.001\,159\,652\,187\,9\,(43)$$

H. Dehmelt et al. 1987

$$a_e = 0.001\,159\,652\,180\,85\,(76)$$

G. Gabrielse et al. 2006

$$a_e = \frac{\alpha}{2\pi} - 0.328\ldots\left(\frac{\alpha}{\pi}\right)^2 + 1.182\ldots\left(\frac{\alpha}{\pi}\right)^3 - 1.505\ldots\left(\frac{\alpha}{\pi}\right)^4$$

Theory

$$a_e = 0.001\,159\,652\,133\,(290)$$

$$a_e = 0.001\,159\,652\,180\,85\,(76)$$

4. Experimental determination of muon g-2

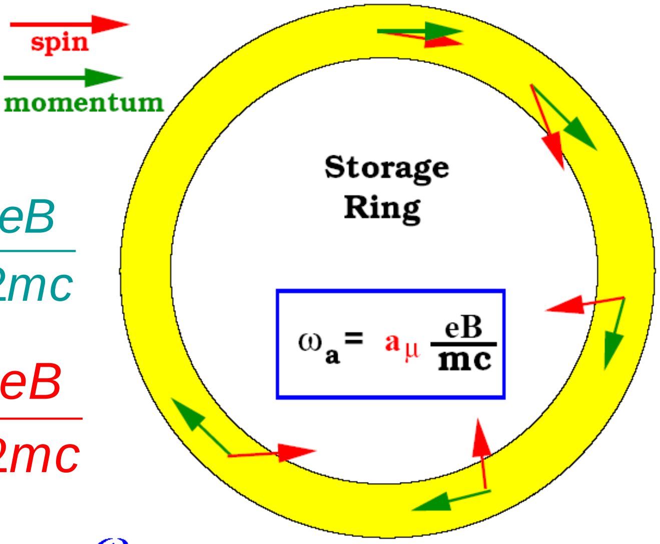
Principle:

- store polarized muons in a storage ring; revolution with cyclotron frequency ω_c
- measure spin precession around the magnetic dipole field relative to the direction of cyclotron motion

$$\omega_C = 2\frac{eB}{2mc}$$

$$\omega_S = g\frac{eB}{2mc}$$

$$\omega_a = \omega_S - \omega_C$$



(exaggerated ~20x)

Precession:

$$\vec{\omega}_a = -\frac{e}{m_\mu c}\left[a_\mu \vec{B} - (a_\mu - \frac{1}{\gamma^2 - 1})\vec{\beta}\times\vec{E}\right]$$

Difference between Lamor and cyclotron frequency

Effect of electrical focussing fields (relativistic effect).

$= 0$ for $\gamma = 29.3$

$\Leftrightarrow p_\mu = 3.094\,\mathrm{GeV/c}$

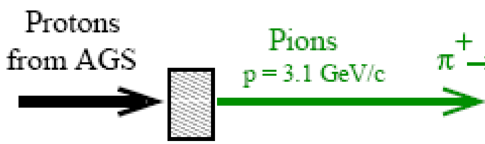
First measurements:

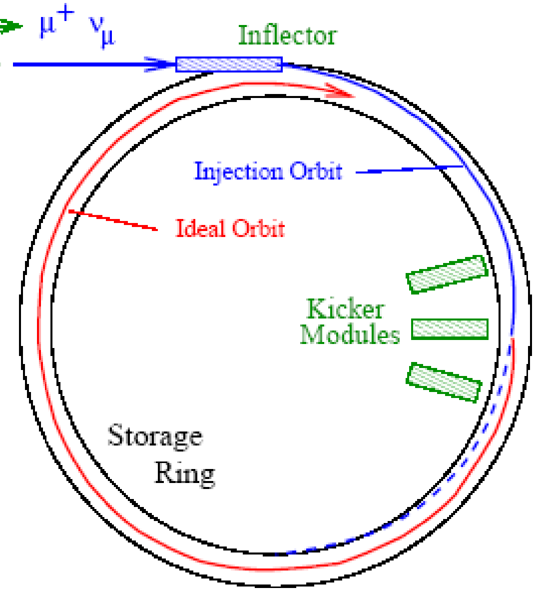
CERN 70s

$a_{\mu^-} = 0.001165937(12)$

$a_{\mu^+} = 0.001165911(11)$

(g-2)$_\mu$ Experiment at BNL

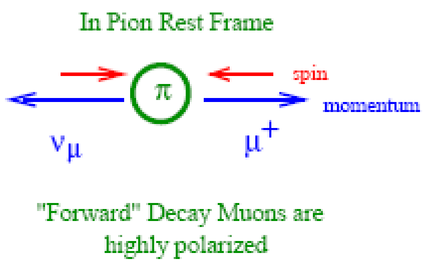


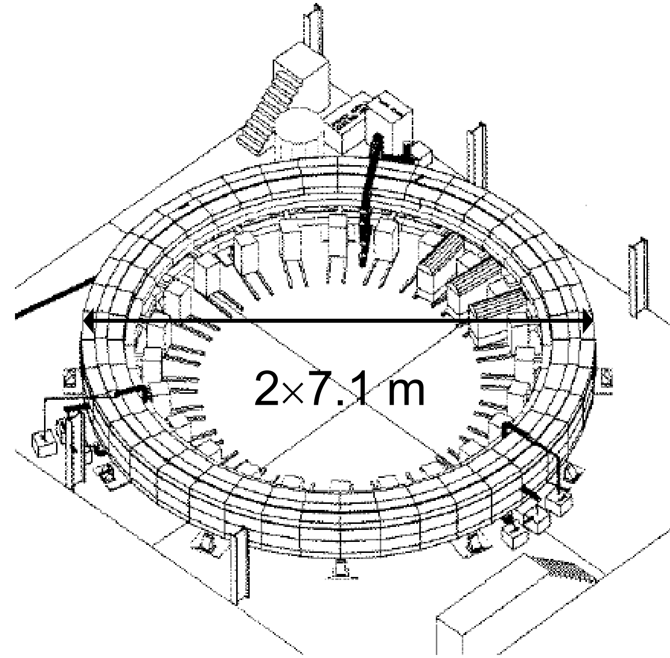


E=24GeV Target

1 μ / 10^9 protons on target

6x10^{13} protons / 2.5 sec

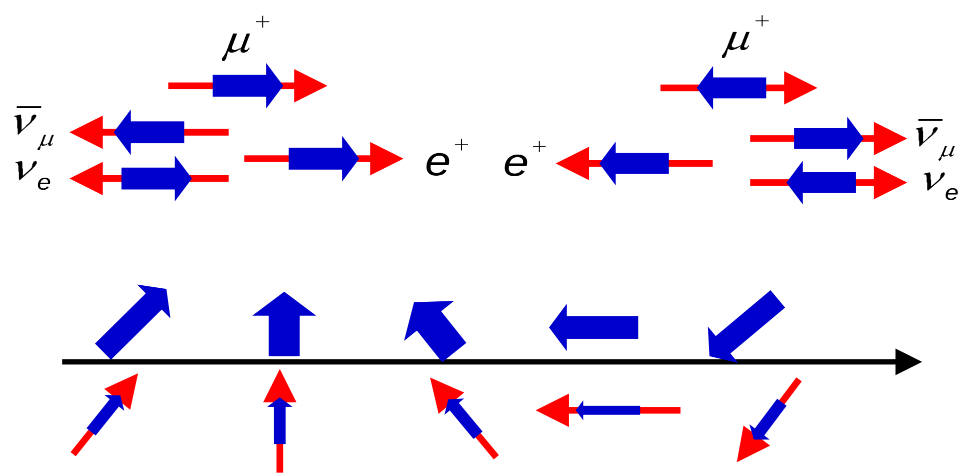




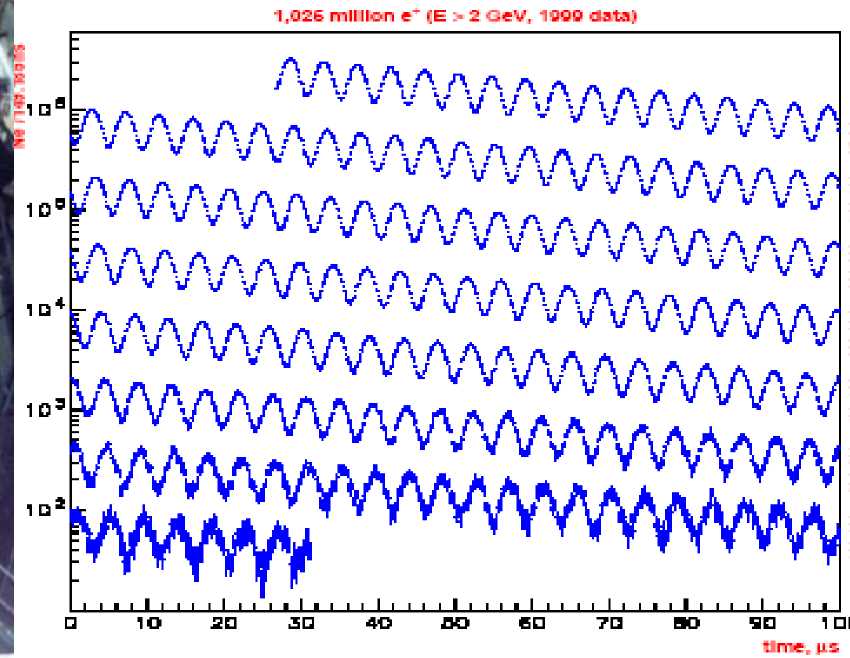
"V-A" structure of weak decay:

Use high-energy e^+ from muon decay to measure the muon polarization

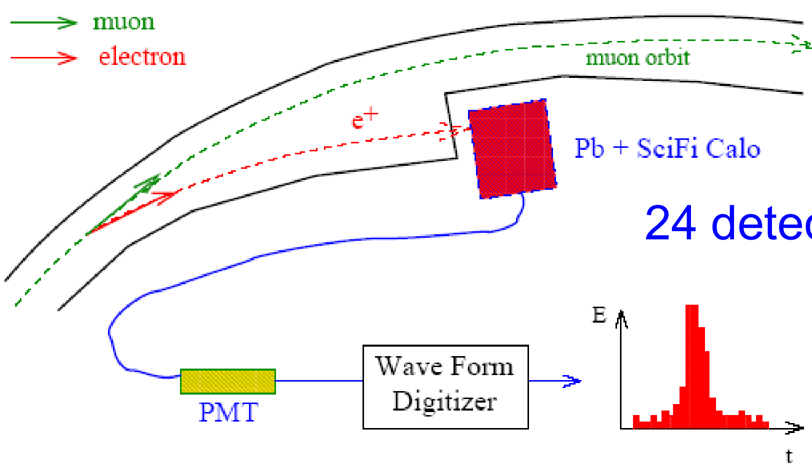
Weak charged current couples to LH fermions (RH anti-fermions)



Measure electron rate:



$$N(t) = N_0\, e^{-\lambda t} \left[1 + A \cos(\omega_a t + \varphi) \right]$$



24 detectors

$$\frac{\omega_a}{2\pi} = 229023.59(16)\,\text{Hz}$$

(0.7ppm)

$$a_\mu = \frac{\omega_a}{\frac{e}{m_\mu c}\langle B \rangle} \quad ?$$

From ω_a to a_μ - How to measure the B field

$<B>$ is determined by measuring the proton nuclear magnetic resonance (NMR) frequency ω_p in the magnetic field.

$$a_\mu = \frac{\omega_a}{\frac{e}{m_\mu c}\langle B\rangle} = \frac{\omega_a}{\frac{e}{m_\mu c}\frac{\hbar\tilde{\omega}_p}{2\mu_p}} = \frac{\omega_a}{\frac{4\mu_\mu}{\hbar g_\mu}\frac{\hbar\tilde{\omega}_p}{2\mu_p}} = \frac{\omega_a/\tilde{\omega}_p}{\mu_\mu/\mu_p}(1+a_\mu)$$

$$\Downarrow$$

$$a_\mu = \frac{\omega_a/\omega_p}{\mu_\mu/\mu_p - \omega_a/\omega_p}$$

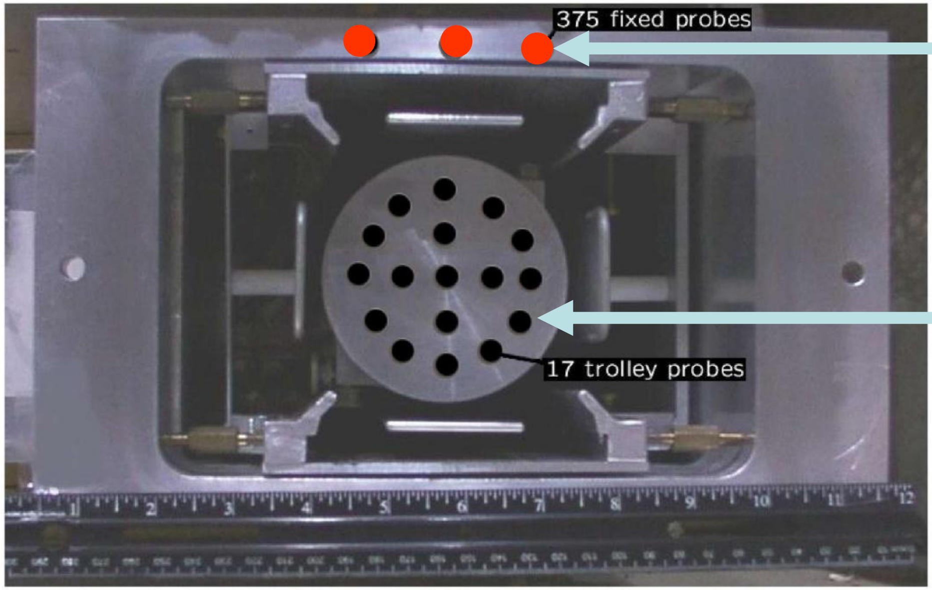
Frequencies can be measured very precisely

$\mu_{\mu+}/\mu_p = 3.183\ 345\ 39(10)$

from hyperfine splitting in muonium

W. Liu *et al.*, Phys. Rev. Lett. **82**, 711 (1999).

NMR trolley

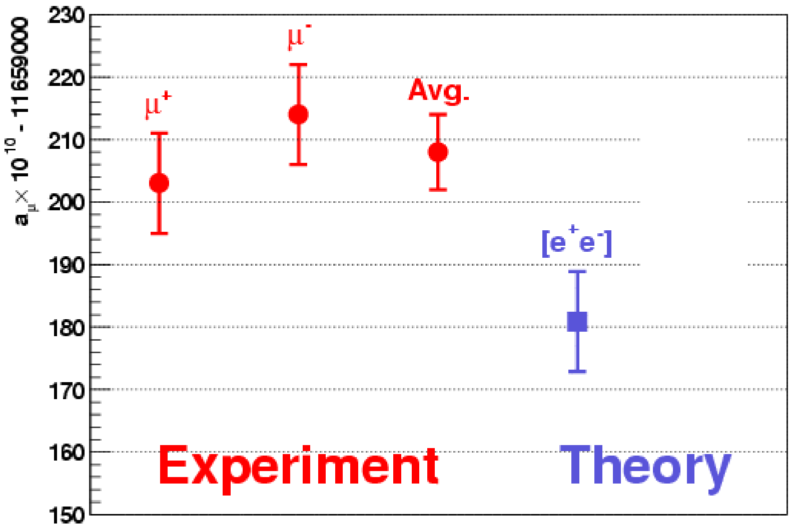


375 fixed NMR probes around the ring

17 trolley NMR probes

$\tilde{\omega}_p/2\pi$ = 61 791 400(11) Hz (0.2ppm)

$$a_{\mu^+} = 11\,659\,203(8)\times 10^{-10}(0.7ppm)$$
$$a_{\mu^-} = 11\,659\,214(8)\times 10^{-10}(0.7ppm)$$
$$a_{\mu} = 11\,659\,208(6)\times 10^{-10}(0.5ppm)$$



About 2.6σ deviation:

- Often interpreted as sign of new physics: SUSY
- But careful:
  "Theory" has uncertainties …
  … and sometimes even bugs.
- Quantum loop effects (SM or new physics) are ~ m^2 and therefore more important for muons than for electrons.

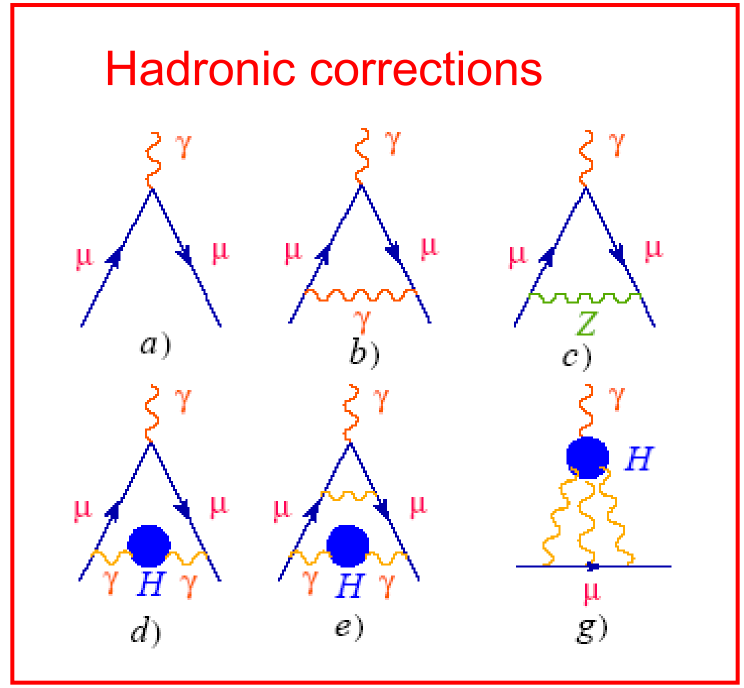
5. Theoretical prediction of a_μ

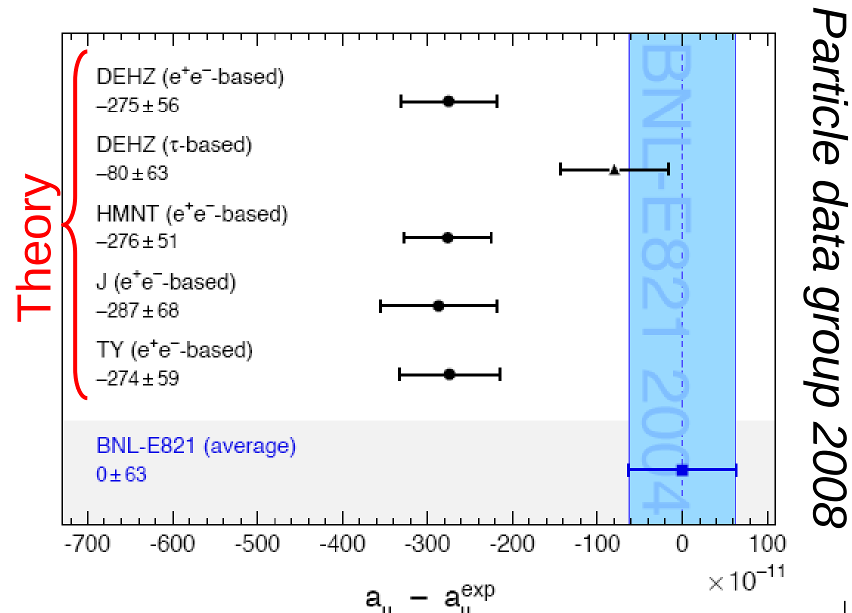
Beside pure QED corrections there are weak corrections (W, Z) exchange and „hadronic corrections"

$$a_\mu = a_\mu^{QED} + a_\mu^{Had} + a_\mu^{EW}$$

(For the electron with much lower mass the hadronic and weak corrections are suppressed (~m²), and can be neglected.)

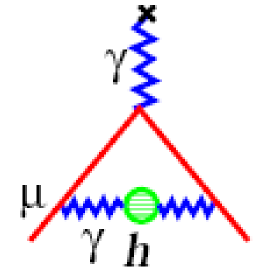
→ Determination of hadronic corrections is difficult and is in addition based on data: hot discussion amongst theoreticians how to correctly use the data.



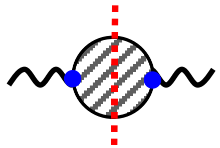
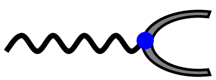


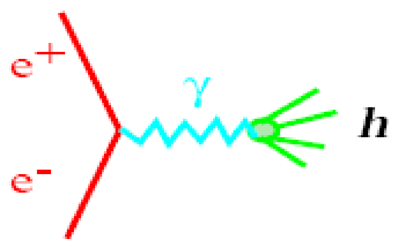
Hadronic vacuum polarization:

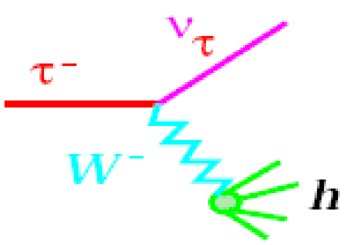
Hadronic corrections related to virtual intermediate hadronic states ($\pi\pi$, ρ, ϕ) – cannot be calculated.



Use the "optical theorem" to relate the loop corrections to observable cross sections / branching ratios:

Im[] $\propto$ |  hadrons |^2

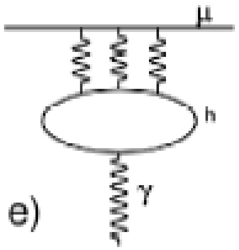




$$a_\mu(\mathrm{had};1) = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4m_\pi^2}^{\infty} \frac{ds}{s^2} K(s)\, R(s)$$

… calculations are sometimes not easy …

In 2001 Kinoshita et al. found a sign mistake in their calculation of the light-by-light scattering amplitude:



e)

December 2001
KEK-TH-793
hep-ph/0112102

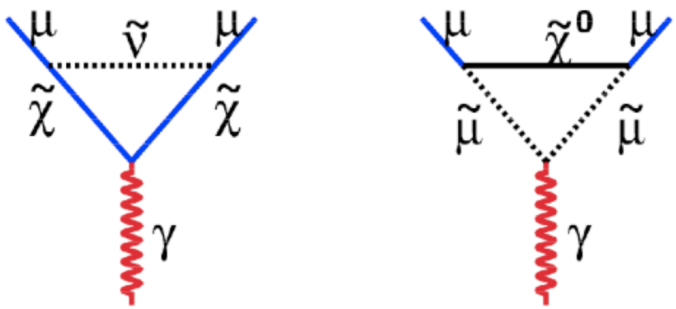
Comment on the sign of the pseudoscalar pole contribution to the muon $g-2$

Masashi Hayakawa * and Toichiro Kinoshita †

Abstract

We correct the error in the sign of the pseudoscalar pole contribution to the muon $g-2$, which dominates the $\mathcal{O}(\alpha^3)$ hadronic light-by-light scattering effect. The error originates from our oversight of a feature of the algebraic manipulation program FORM which defines the ϵ-tensor in such a way that it satisfies the relation $\epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\nu_1\nu_2\nu_3\nu_4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3}\eta^{\mu_4\nu_4} = 24$, irrespective of space-time metric. To circumvent this problem, we replaced the product $\epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\nu_1\nu_2\nu_3\nu_4}$ by $-\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}\eta_{\mu_4\nu_4} \pm \cdots$ in the FORM-formatted program, and obtained a positive value for the pseudoscalar pole contribution, in agreement with the recent result obtained by Knecht *et al.*

Potential SUSY contribution to muon (g-2)



Potential SUSY contributions:

For muon ~40000 times larger than in case of electrons.

$$a_\mu^{\rm SUSY} \simeq \pm 130 \times 10^{-11} \cdot \left(\frac{100\ {\rm GeV}}{m_{\rm SUSY}}\right)^2 \tan\beta\,,$$

$$a_\mu = a_\mu^{QED} + a_\mu^{Had} + a_\mu^{EW} + a_\mu^{SUSY}$$

First sign of New Physics ??