QGP Physics − From Fixed-Target to LHC

4. Thermodynamics of the QGP

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4.1 QGP in the MIT bag model

Thermodynamics of
  Relativistic Bose gas
  Relativistic Fermi-gas
Bag model of hadrons
Constructing the phase diagram between pion gas and QGP
“ realistic hadron gas and QGP

4.1.1 Thermodynamics of a relativistic Bose gas

probability density for occupation of state with relativistic energy E and degeneracy g

$$\mathrm{N(E)} = \frac{\mathrm{g}}{(2\pi)^3}(\exp(\frac{\mathrm{E}-\mu}{\mathrm{T}})-1)^{-1}$$

with energy $\mathrm{E}^2 = \mathrm{p}^2 + \mathrm{m}^2$

(note: here $\hbar$=c=1)

and chemical potential μ

neglecting the particle mass (okay since in interesting region $\mathrm{E} = 3\mathrm{T} \gg \mathrm{m}$)
and chemical potential (good as long as no additive quantum number)

Boson number density

$$\mathrm{n} = \int \mathrm{N(E)d^3p} = \frac{4\pi \mathrm{g}}{(2\pi)^3}\int \frac{\mathrm{p^2dp}}{\exp(\frac{\mathrm{p}}{\mathrm{T}})-1}$$

$$\boxed{\mathrm{n} = \frac{\mathrm{g}}{\pi^2}\mathrm{T}^3\zeta(3)}$$

with Rieman ζ-function $\zeta(3) \approx 1.2$

Thermodynamics of a relativistic Bose gas

Boson energy density $$\epsilon = \int \mathrm{N(E)pd^3p} = \frac{4\pi \mathrm{g}}{(2\pi)^3} \int \frac{\mathrm{p^3dp}}{\exp(\frac{\mathrm{p}}{\mathrm{T}}) - 1}$$

$$\epsilon = \frac{3\mathrm{g}}{\pi^2} \mathrm{T}^4 \zeta(4)$$

with Rieman ζ-function $\zeta(4) = \frac{\pi^4}{90} \approx 1.08$

$$\epsilon = \frac{\pi^2}{30} \mathrm{g} \mathrm{T}^4$$

and we get the Energy per particle $\epsilon/\mathrm{n} = 3\mathrm{T}\frac{\zeta(4)}{\zeta(3)} \approx 2.7\,\mathrm{T}$

Boson pressure $\mathrm{P} = \mathrm{n}^2 \partial \frac{\epsilon}{\mathrm{n}} / \partial \mathrm{n} \rightarrow \mathrm{P} = \frac{1}{3}\epsilon$

Entropy density $\mathrm{d}\sigma = \mathrm{d}\epsilon/\mathrm{T}$ and $\mathrm{d}\epsilon = \mathrm{const.\,T^3dT}$

$$\sigma = \int \mathrm{d}\sigma = \mathrm{const.} \int \mathrm{T^2dT} = \frac{1}{3} \mathrm{const.\,T^3} \quad \rightarrow \quad \sigma = \frac{1}{3}\frac{\mathrm{d}\epsilon}{\mathrm{dT}}$$

$$\sigma = \frac{4\pi^2}{90}\,\mathrm{g\,T^3}$$

Thermodynamics of a relativistic Bose gas

and the entropy per particle (boson) $\sigma/\mathrm{n} = 4\zeta(4)/\zeta(3) \approx 3.6$

old Landau formula for pions: S = 3.6 dN/dy

4.1.2 Thermodynamics of a relativistic Fermi gas

probability density for occupation $\mathrm{N(E)} = \frac{g}{(2\pi)^3}(\exp(\frac{E-\mu}{T})+1)^{-1}$

but now μ not generally 0

Fermion number density $\mathrm{n} = \frac{4\pi g}{(2\pi)^3}\int \frac{p^2 dp}{\exp(\frac{p-\mu}{T})+1}$

Energy density $\epsilon = \frac{4\pi g}{(2\pi)^3}\int \frac{p^3 dp}{\exp(\frac{p-\mu}{T})+1}$

cannot be solved analytically, only numerically

but there is analytic solution for sum of particle and antiparticle (e.g. quark and antiquark) (Chin, PL 78B (1978) 552)

and $$\epsilon_q + \epsilon_{\bar{q}} = g(\frac{7\pi^2}{120}T^4 + \frac{\mu^2}{4}T^2 + \frac{\mu^4}{8\pi^2})$$

$$n_q - n_{\bar{q}} = g(\frac{\mu}{6}T^2 + \frac{\mu^3}{6\pi^2})$$

specific example for fermions: quarks in QGP with no net baryon density

$$\langle q \rangle = \langle \bar{q} \rangle \leftrightarrow \mu = 0$$

in that case quark number density $\mathrm{n_q = \frac{g}{\pi^2} T^3 d(3)}$

Note: $\mathrm{d(\alpha + 2) = \int \frac{x^\alpha dx}{e^x + 1}}$ and $\mathrm{d(3) \approx 0.9}$

and quark and antiquark energy density $\mathrm{\epsilon_q = \epsilon_{\bar{q}} = \frac{3g}{\pi^2} T^4 \, d(4) = \frac{7\pi^2}{240} \, g \, T^4}$

with $\mathrm{d(4) = \frac{7\pi^4}{720}}$

the energy per quark is then $\mathrm{\epsilon/n = 3T \frac{d(4)}{d(3)} \approx 3.2T}$

Entropy density (computed as above for bosons)

$$\mathrm{\sigma = \frac{7\pi^2}{180} \, g \, T^3}$$

and the entropy per fermion (quark) $\mathrm{\sigma/n = 4 \frac{d(4)}{d(3)} \approx 4.2}$

Summary relativistic bosons and fermions (no chem.pot.)

- Energy density $\epsilon \propto \mathrm{T}^4$
- Pressure $\mathrm{P} = \frac{1}{3}\epsilon \propto \mathrm{T}^4$
- Entropy density $\sigma \propto \mathrm{T}^3$
- Particle number density $\mathrm{n} \propto \mathrm{T}^3$

→ to obtain physical units of GeV/fm3 or fm-3, multiply with appropriate powers of ħc

- all are proportional to the number of degrees of freedom
- between bosons and fermions there is a factor 7/8

$$\epsilon_\mathrm{f} = \frac{7}{8}\epsilon_\mathrm{b} \quad \text{etc.}$$

4.1.3 Short excursion: the bag model

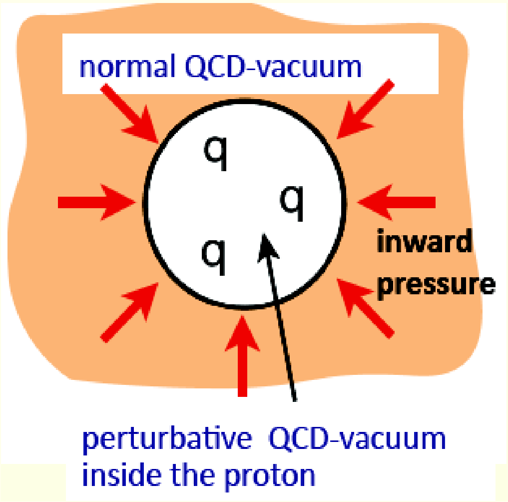
to deal with QCD in the nonperturbative regime (i.e. where α_s is not negligible) one needs to make models (alternative: lattice QCD see below)
for instance to treat the nucleon and its excitations

MIT bag model: build confinement and asymptotic freedom into simple phenomenological model

A. Chodos, R.L. Jaffe, K. Johnson, C.B.Thorne, Phys. Rev. D10 (1974) 2599
T. DeGrand, R.L. Jaffe, K. Johnson, J. Kiskis, Phys. Rev. D12 (1975) 2060

hadrons considered as bags embedded into a non-perturbative QCD vacuum also called "physical vacuum" or "normal QCD vacuum"

space divided into 2 regions



Interior of bag: quarks have very small (current) masses, interaction weak

Exterior of bag: quarks are not allowed to propagate there, lower vacuum energy, no colored quarks or gluons but quark and gluon condensates

Hadrons in MIT bag model

Hadrons are considered drops of another, perturbative phase of QCD immersed into normal QCD vacuum

all non-perturbative physics included in one universal quantity, the bag constant B defined as the difference in energy density between perturbative and physical vacua:

$$\epsilon_{\mathrm{bag}} - \epsilon_{\mathrm{vac}} \equiv \mathrm{B} > 0$$

solve Dirac equation for massless quarks inside bag with volume V and surface S with special boundary conditions at the surface that

i) enforce confinement: quark current normal to bag surface = 0

ii) define a stability condition for bag: pressure of Dirac particles inside is balanced by difference in energy density inside and outside

$$\mathrm{H} = \mathrm{H}_{\mathrm{kin}} + \mathrm{H}_{\mathrm{spin-spin}} + \mathrm{B\,V}$$

- $\mathrm{H}_{\mathrm{kin}}$ → kin. Energy of quarks confined in bag
- $\mathrm{H}_{\mathrm{spin-spin}}$ → spin-spin interaction
- $\mathrm{B\,V}$ → energy to make hole of volume V in phys. vacuum

Hadrons in MIT bag model

for (nearly) massless quarks $E_{kin} \propto 1/R$ → tries to extend bag
(spherical bag with radius R)

bag term $B\,\frac{4\pi}{3}R^3$ → tries to contract bag

⇨ equilibrium is reached

obtain e.g. for nucleon mass (spherical bag with 3 quarks in s-state)

$$E = 3\frac{\omega_{n,-1}}{R} + \frac{4\pi}{3}BR^3$$ with $\omega_{1,-1} = 2.04$ $\omega_{2,-1} = 5.40$

and $$\frac{\partial E}{\partial R} = 0$$

internal energy determines the radius of the bag, if B is a universal constant

- determines masses and sizes of all hadrons
  rather successful with $B_{MIT} = 56\,\mathrm{MeV/fm^3}$
  baryon octet and decuplet as well as vector mesons well reproduced

note: often instead of B, $B^{1/4}$ in MeV is quoted $B_{MIT}^{1/4} = 146\,\mathrm{MeV}$

4.1.4 Thermodynamics of pion gas and QGP

pion gas: massless bosons with degeneracy $g_\pi = 3 \;\; \text{for} \;\; \pi^+, \pi^0, \pi^-$

energy density of pion gas $\epsilon_\pi = \frac{\pi^2}{30} g_\pi T^4 = 129\, T^4$ and pressure $P = \frac{1}{3}\epsilon = 43\, T^4$

after properly inserting missing powers of $\hbar$c and using T in GeV

quark-gluon plasma:

gluons as massless bosons with degeneracy g_g = 2(spin) x 8(color) = 16
quarks massless fermions with degeneracy g_q = N_f x 2(spin) x 3(color) = 6 N_f
and same for antiquarks (here N_f is number of massless/light flavors)

additional contribution to energy density: to make quark-gluon gas, need to create cavity in vacuum

energy needed is given by the bag constant B "pressure of vacuum on color field"

analogy to Meissner effect: superconductor expells magnetic field

$\leftrightarrow$ QCD vacuum expels color field into bags

$\rightarrow \quad \epsilon = \epsilon_{\text{thermal}} + B$ and deriving pressure as above $P = \frac{1}{3}(\epsilon - 4B)$

4.1.4 Thermodynamics of pion gas and QGP

What value to use for the bag constant?
from hadron phenomenology at T=0 and normal nuclear matter density

$$\mathrm{B} \approx 50 - 100\,\mathrm{MeV/fm^3}$$

but there are a number of problems with MIT bag model

and there is good indication that B derived there is not the energy density of the QCD vacuum; conclusion: hadrons are not small drops of the new QCD phase but only a relatively small perturbation of the QCD vacuum

also B = B(T,n) (see e.g. Shuryak, the QCD vacuum...)
basic argument: at large T,n all non-perturbative phenomena suppressed

$\mathrm{B_{eff}} \approx 500 - 1000\,\mathrm{MeV/fm^3}$ vacuum energy density

energy density of quark-gluon gas

$$\epsilon_{\mathrm{qg}} = \frac{\pi^2}{30}(\mathrm{g_g} + \frac{7}{8}\mathrm{g_q})\mathrm{T}^4 + \mathrm{B} = \frac{\pi^2}{30}(16 + \frac{21}{2}\mathrm{N_f})\mathrm{T}^4 + \mathrm{B}$$

for N_f = 2 (u,d)

$$\epsilon_{\mathrm{qg}} = 1592\,\mathrm{T}^4 + 0.5 \quad (\frac{\mathrm{GeV}}{\mathrm{fm}^3})$$

Constructing the phase diagram

system always in phase with highest pressure

Gibbs conditions for critical point

$$\mathrm{P_{QGP}} = \mathrm{P_{piongas}}$$

and

$$\mu_{\mathrm{QGP}} = \mu_{\mathrm{piongas}} (= 0)$$

for $N_f = 2$

$$\frac{3\pi^2}{90}\mathrm{T_c^4} = \frac{\pi^2}{90}(16 + \frac{21}{2}\mathrm{N_f})\mathrm{T}^4 - \mathrm{B}$$

$$\frac{34\pi^2}{90}\mathrm{T_c^4} = \mathrm{B} \qquad \mathrm{T_c} = (\frac{90 \cdot 0.5\mathrm{GeV} \cdot 0.197^3\mathrm{GeV^3 fm^3}}{34\pi^2 \mathrm{fm^3}})^{1/4} = 0.18\,\mathrm{GeV}$$

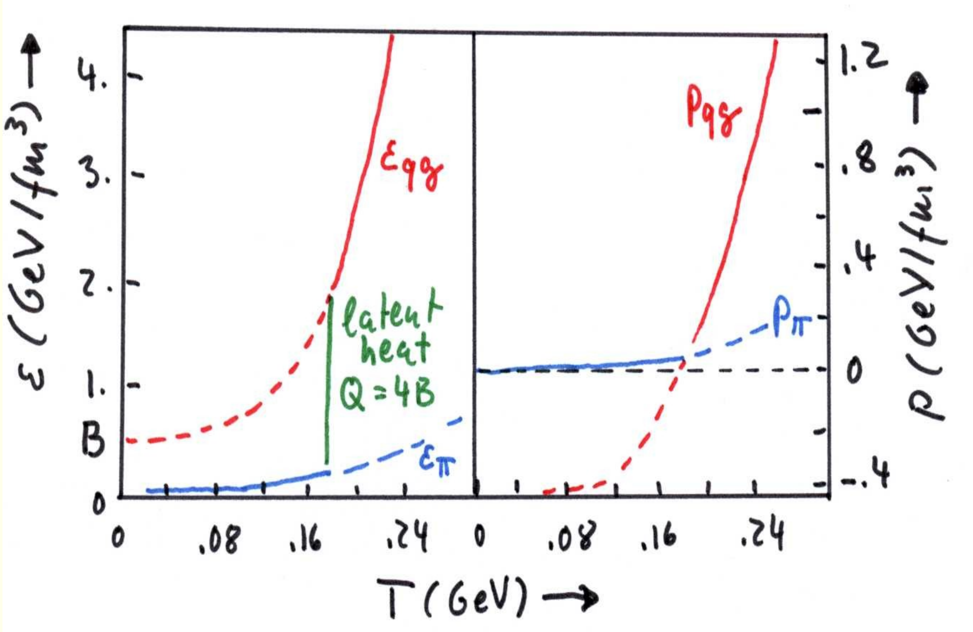
latent heat:

$$\epsilon_{\mathrm{qg}} - \epsilon_{\mathrm{pion}}(\mathrm{at T_c}) = \frac{34\pi^2}{30}\mathrm{T_c^4} + \mathrm{B}$$

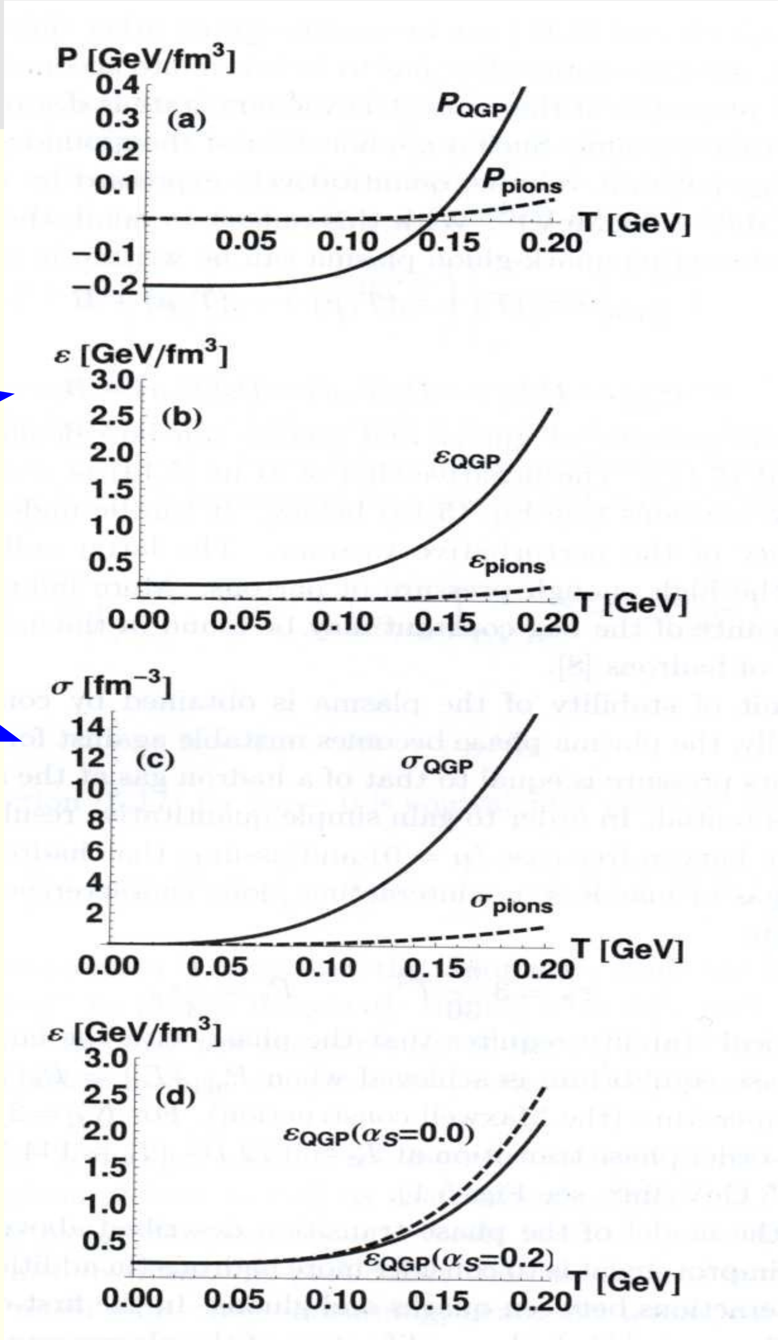
$$= 1.54 + 0.5 = 2\frac{\mathrm{GeV}}{\mathrm{fm^3}}$$

change in entropy density:

$$\sigma_{\mathrm{qg}} - \sigma_{\mathrm{pion}} = \frac{34 \cdot 4\pi^2}{90}\mathrm{T_c^3} = 11.4/\mathrm{fm^3}$$



summary pion gas and QGP at zero baryon chemical potential



pressure
energy density
entropy density

for a gas of massless pions and an ideal quark-gluon plasma with 2 quark flavors

(note: here $B^{1/4}$=200 MeV or $B=0.209\ \mathrm{GeV/fm^3}$ is used)

effect of finite coupling constant in QGP reduces energy density slightly
- first order perturbation theory

S.A.Chin Phys. Lett. B78 (1978) 552

Now check the high baryon density limit

compute a T = 0 μ ≠0 point

cannot do this with pions alone, need nucleons

$$\mathrm{P_{pion}} = 0 \qquad \mathrm{P_{nucleon}} = \frac{\mathrm{g}\mu^4}{3 \cdot 8\pi^2}$$ with g=4 (2(spin) x 2(isospin))

for the quark-gluon side at T=0

$$\mathrm{P_{q\bar{q}}} = \frac{\mathrm{g}\mu^4}{3 \cdot 8\pi^2} - \mathrm{B}$$ with g=12 (quarks and antiquarks)

$$\mathrm{P_{nucleon}} = \mathrm{P_{q\bar{q}}} \quad \rightarrow$$

$$\mu = \left(\frac{3\pi^2 \cdot 0.5\mathrm{GeV} \cdot 0.197^3\mathrm{GeV}^3\mathrm{fm}^3}{\mathrm{fm}^3}\right)^{1/4}$$

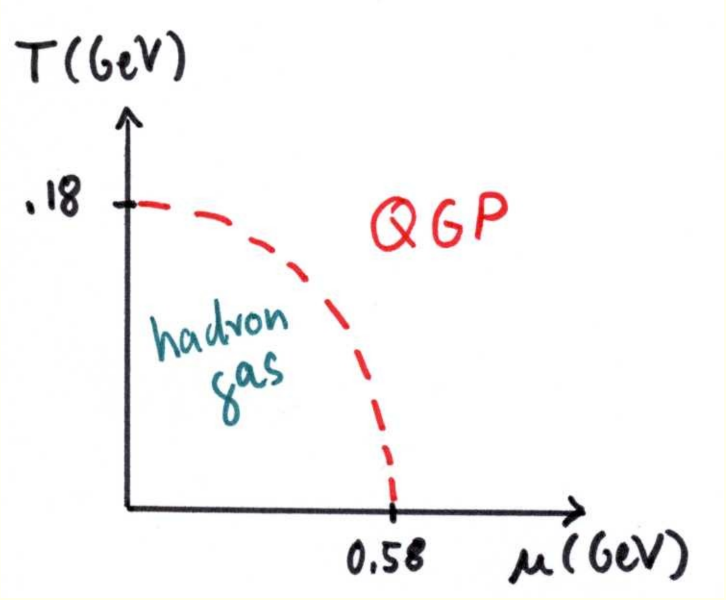
$$= 0.58\,\mathrm{GeV}$$

simple thermodynamik model gives first order phase transition,

Caution: this sets the scale, but there are a number of approximations

pion gas is oversimplification for hadronic matter

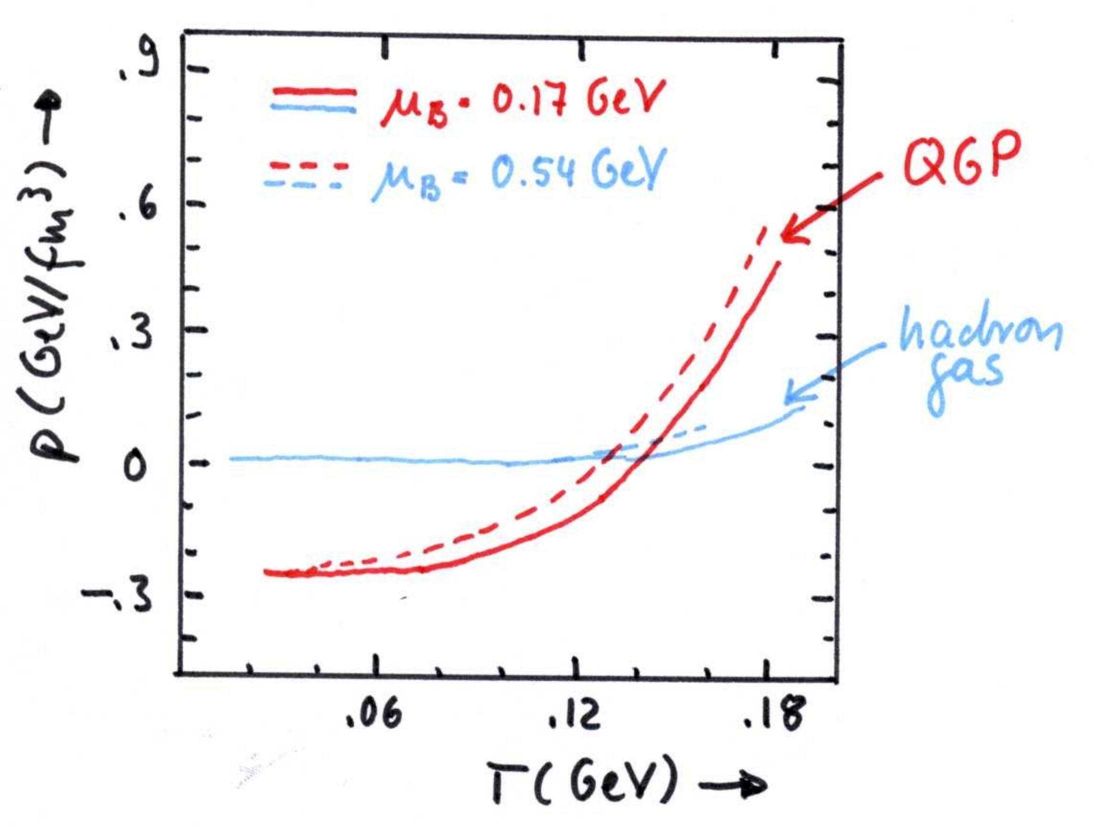
B = 0.5 GeV/fm^3 (should use B(T,n))



4.1.5 more realistic: replace pion gas by hadron gas

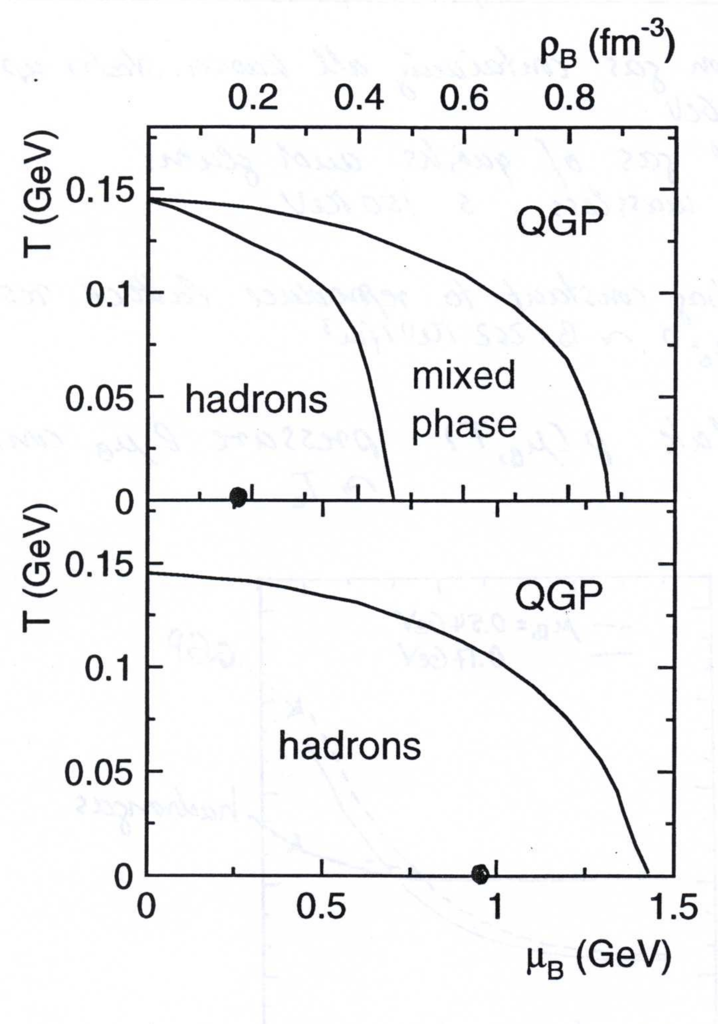
implement all know hadrons up to 2 GeV in mass
ideal gas of quarks and gluons, u,d massless, s 150 MeV
fix bag constant to match lattice QCD result (see below) at $\mu_b=0$ ► $B=262\ \mathrm{MeV/fm^3}$

compute $P(\mu_b,T)$ with
with pressure and μ_b continuous
to obtain T_c

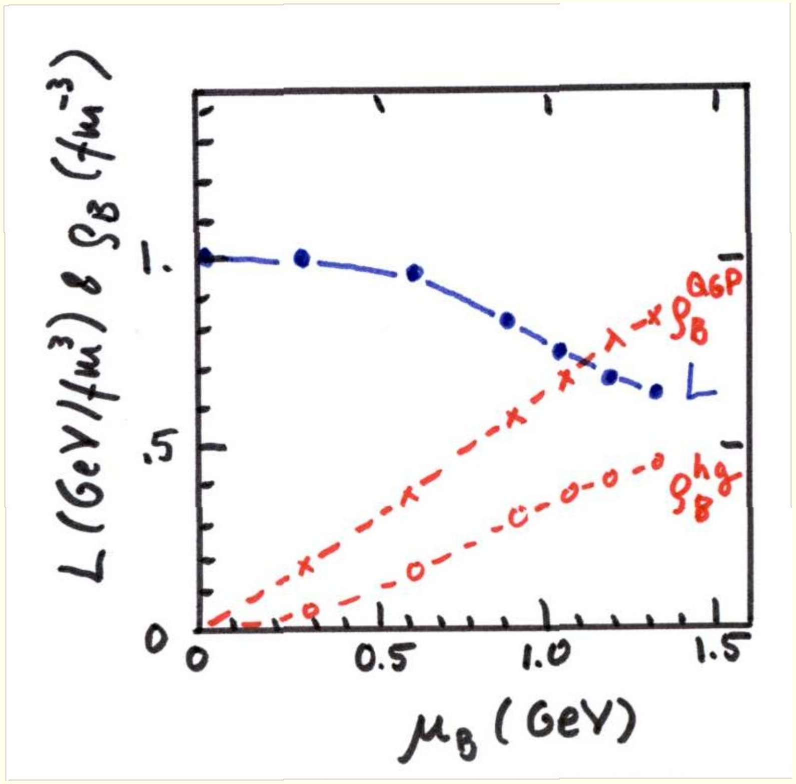


P. Braun-Munzinger, J. Stachel Nucl.Phys. A606 (1996) 320

Phase diagram constructed with hadron gas and QGP



P. Braun-Munzinger, J. Stachel Nucl.Phys. A606 (1996) 320



Note: chemical potential is continuous at phase transition but not the baryon density!

Speed of sound

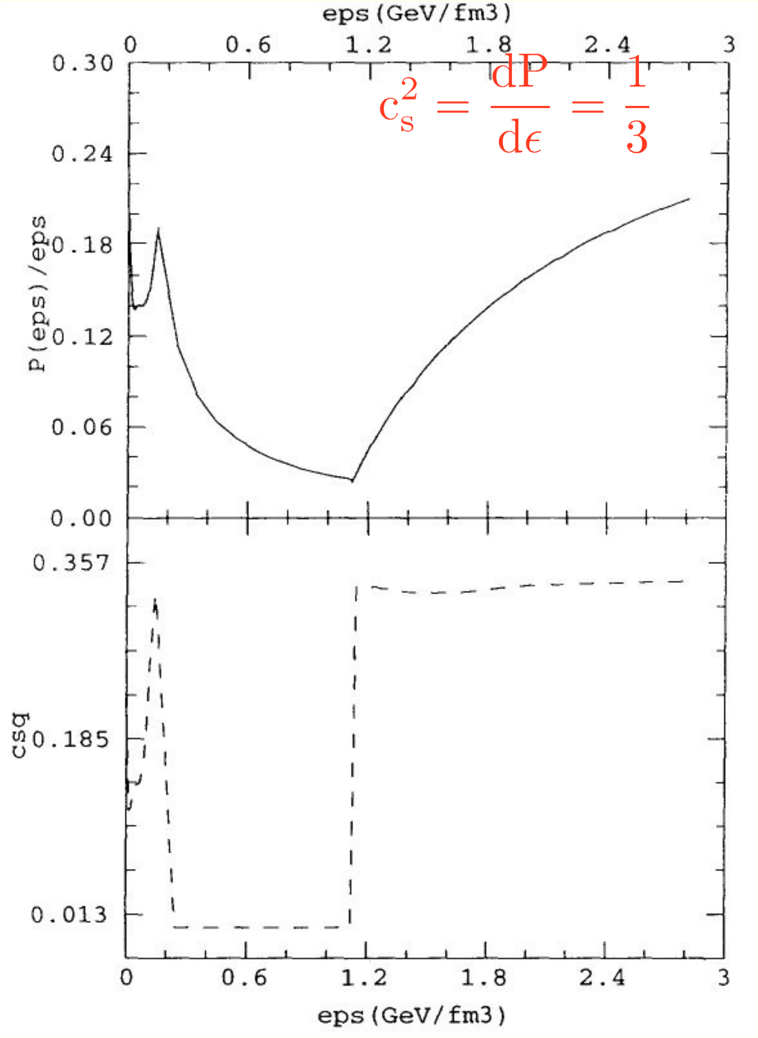
in relativistic gas, speed of sound squared

$$c_s^2 = \frac{dP}{d\epsilon} = \frac{1}{3}$$ without interactions

but in vicinity of phase transition strong deviation of P from 1/3 ε

there is always a minimum in speed of sound

leading to a so-called 'softest point'



P. Braun-Munzinger, J. Stachel Nucl.Phys. A606 (1996) 320

4.2 Lattice QCD

QCD asymptotically free at large T and/or small distances
at low T and for finite size systems $\alpha_s = O(1)$

→ cannot use perturbation theory

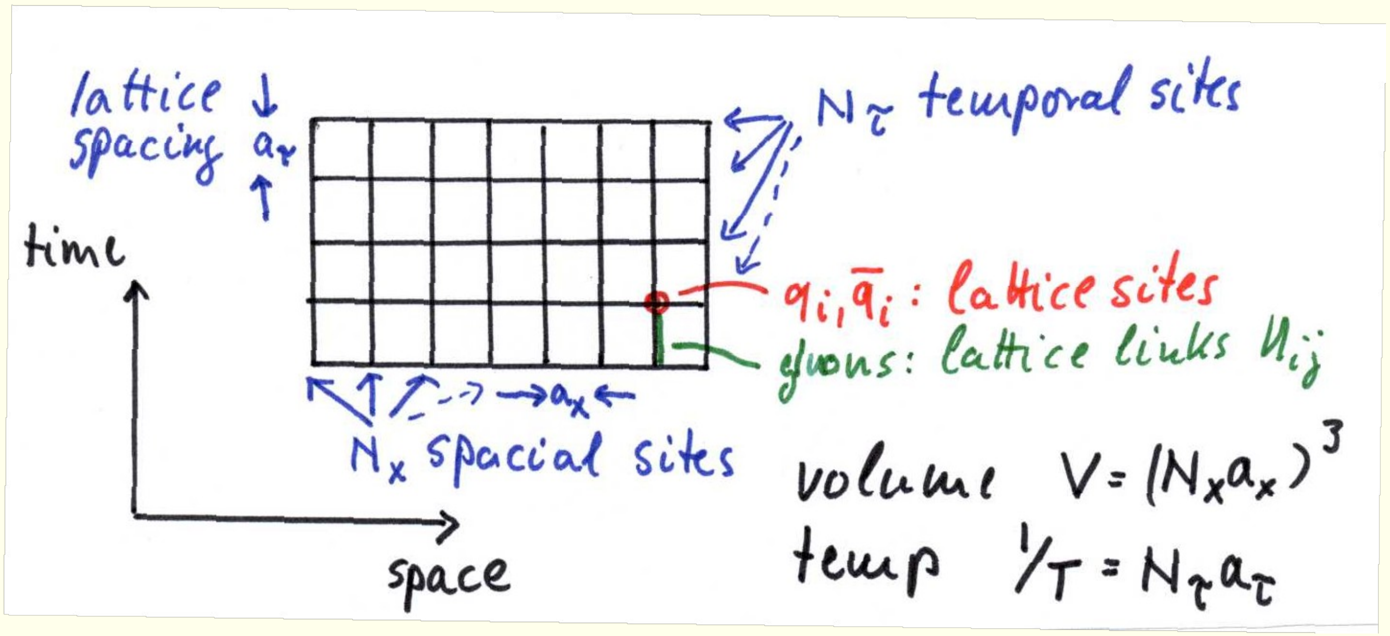
instead solve QCD numerically at zero and finite temperature by putting gauge field on a space-time lattice ↔ "lattice QCD"

some references: M. Creutz 'Quarks, Gluons and Lattices (Cambridge U. Press, Cambridge, 1983)
J. Negele, Proc. NATO Advanced Study Institute, 'Hadrons and Hadronic Matter", eds. D. Vautherin et al. (Plenum, 1990)
Regular proceedings of Lattice conferences
Proc. Lattice 91, Nucl. Phys. B, Proc. Suppl.
" Lattice 92, "
etc.

Lattice QCD - schematic outline of basic (4) steps

i) use evolution in Euclidean time $\tau = \mathrm{it} = 1/\mathrm{T}$
to filter out hadronic ground state and/or evaluate thermal average

ii) replace Euclidean x,τ continuum by finite lattice
field theory with infinite number of degrees of freedom ⟶ finite many body problem
quantum field theory equivalent to classical statistical mechanics
with $$\exp(-\mathrm{iHt}) \rightarrow \exp(-\mathrm{H}\tau) = \exp(-\mathrm{S})$$



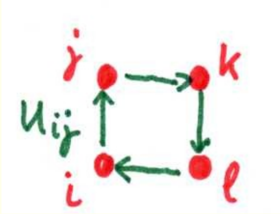
Lattice QCD basic steps

iii) evaluate partition function Z by using Feynman path integrals

$$Z = \mathrm{Tr}\exp(-H_{QCD}\tau)$$

$$\longrightarrow \quad Z = \int \Pi_{links} dU_{ij} \exp(-S(U))$$

action, given by sum over elementary plaquette



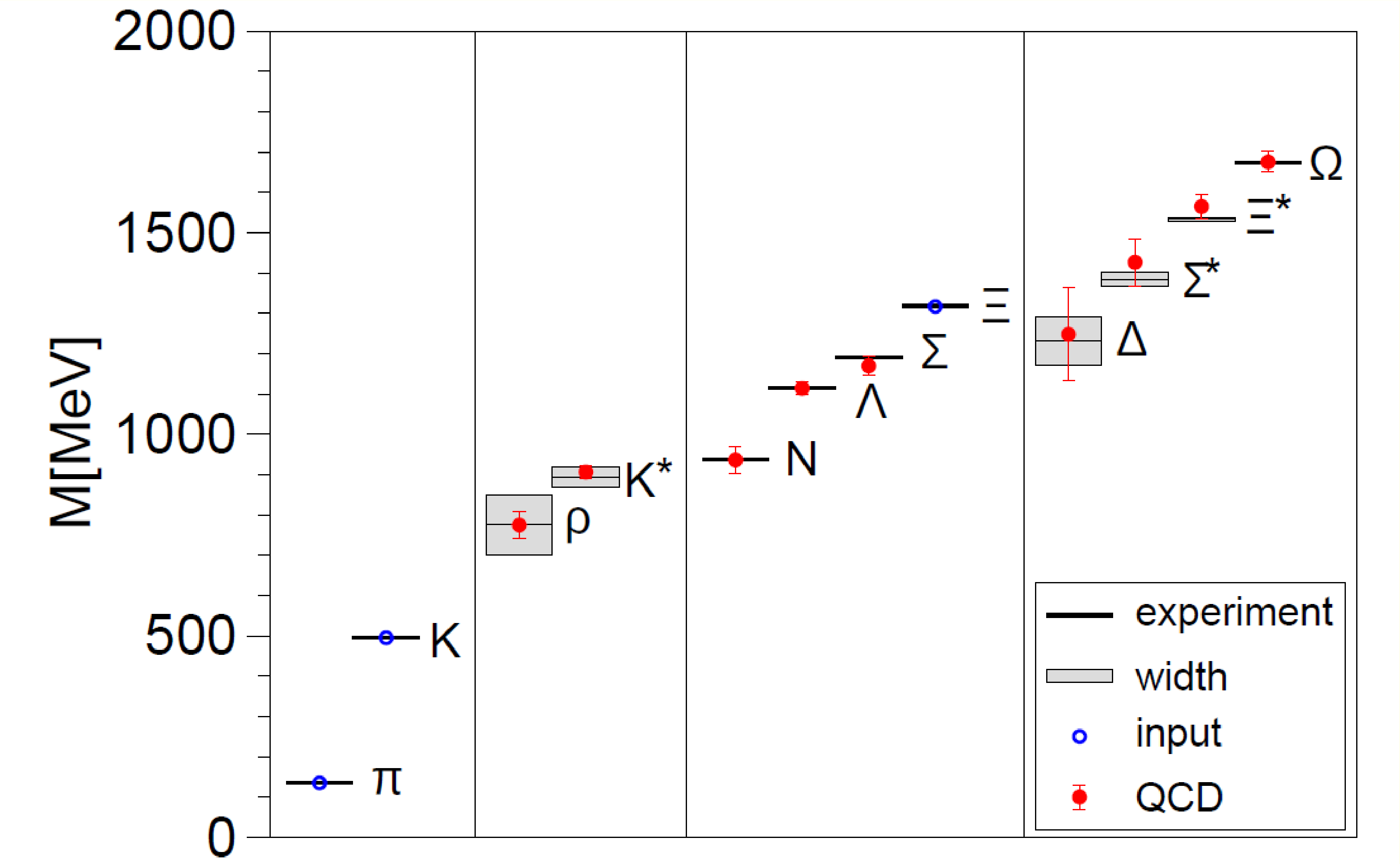
K. Wilson, Phys. Rev. D10 (1974) 2445

iv) lattices need to be big! e.g. 16^3 x 32 sites
have to sum over all color indices at each link $\longrightarrow$ integral 10^7 dimensional
start with some values U_{ij} for all links, successively reassign new elements
to reduce computing time: use stochastic technique with clever weighting (exp(-S(U) favors small action)

have to sweep through entire lattice a few hundred times to evaluate thermodynamic quantities, baryon masses, wave functions

State-of-the-art light hadron spectrum from lattice QCD

S. Dürr, Z.Fodor et al., (Budapest-Marseille–Wuppertal Coll.,Science 322 (2008) 1225



variation of temperature

temperature is changed e.g. by keeping $N\tau$ constant

and changing the lattice spacing

(and thereby the coupling g^2)

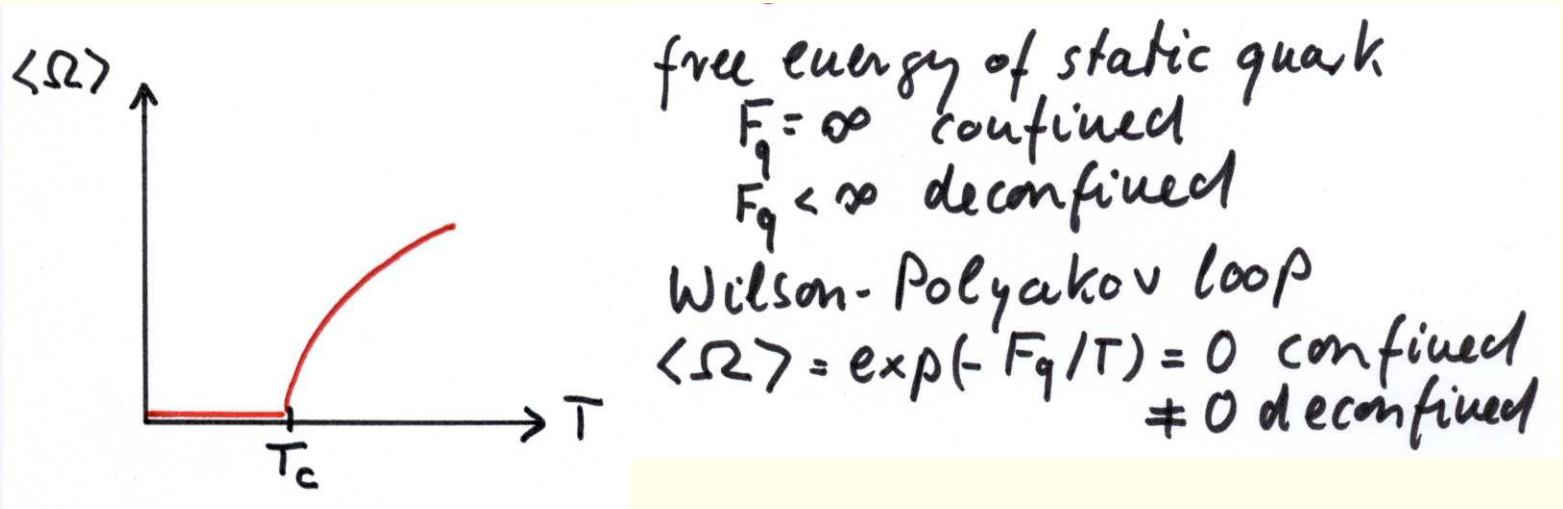
$$a \rightarrow 0, \quad g^2 \rightarrow 0, \quad T \rightarrow \infty$$

approach asymptotic freedom

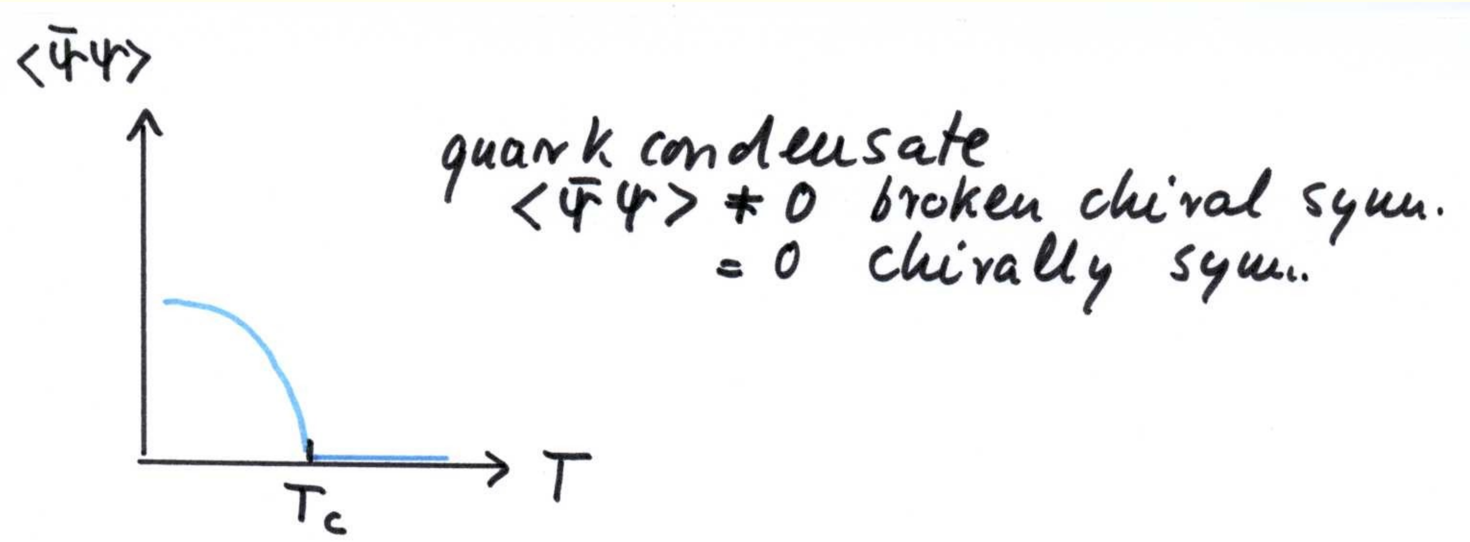
absolute scale is set by calculating a baryon or meson mass in units of a (or g^2)

Indicators of the phase transition

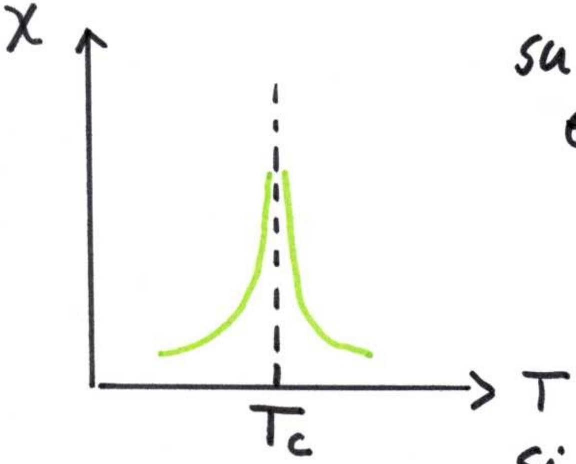
Order parameter of deconfinement:



Order parameter of chiral symmetry restoration:



Indicators of the phase transition



susceptibilities χ

e.g. chiral susc.

$$\chi_m = T \frac{\partial \langle \bar{\Psi} \Psi \rangle}{\partial m}$$

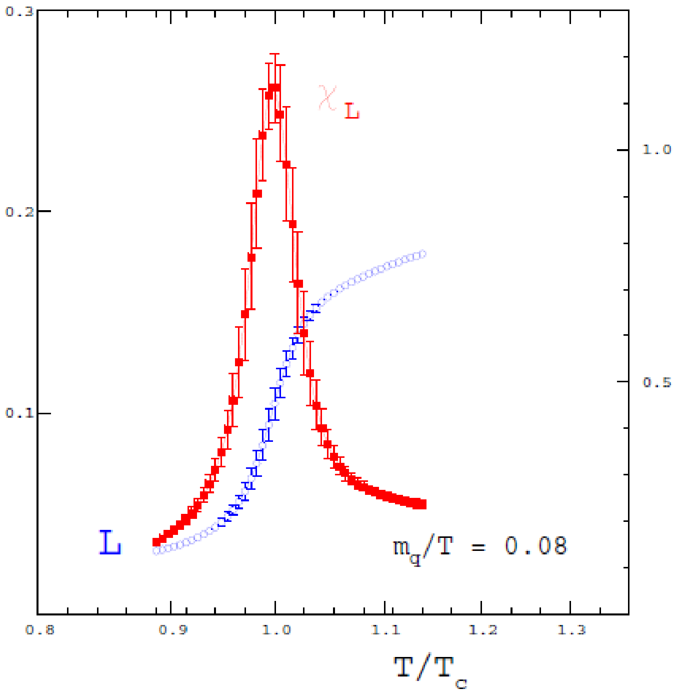
response of quark condens. to mass

similarly Polyakov susceptibility

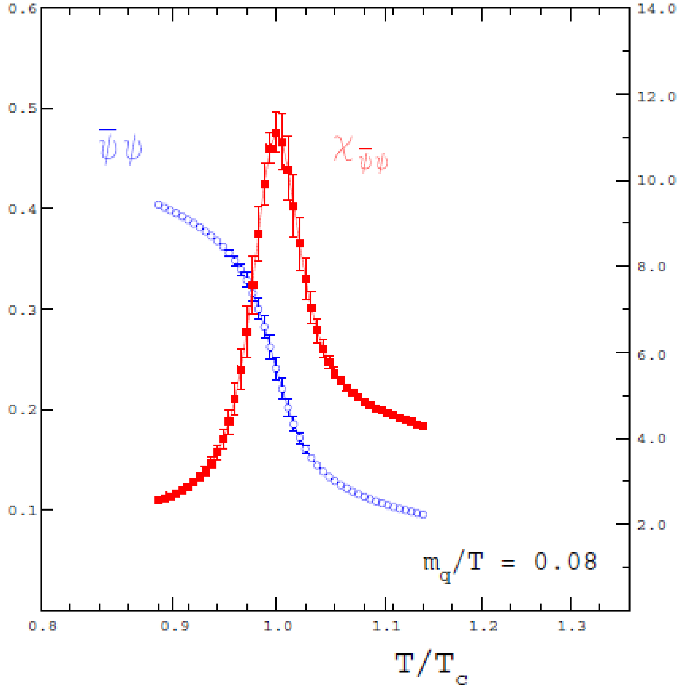
response to addition of a quark

Deconfinement and chiral phase transition in lattice QCD

Polyakov loop - deconfinement



quark condensate $\langle\bar\psi\psi\rangle = \partial p/\partial m$ - chiral sym. restoration

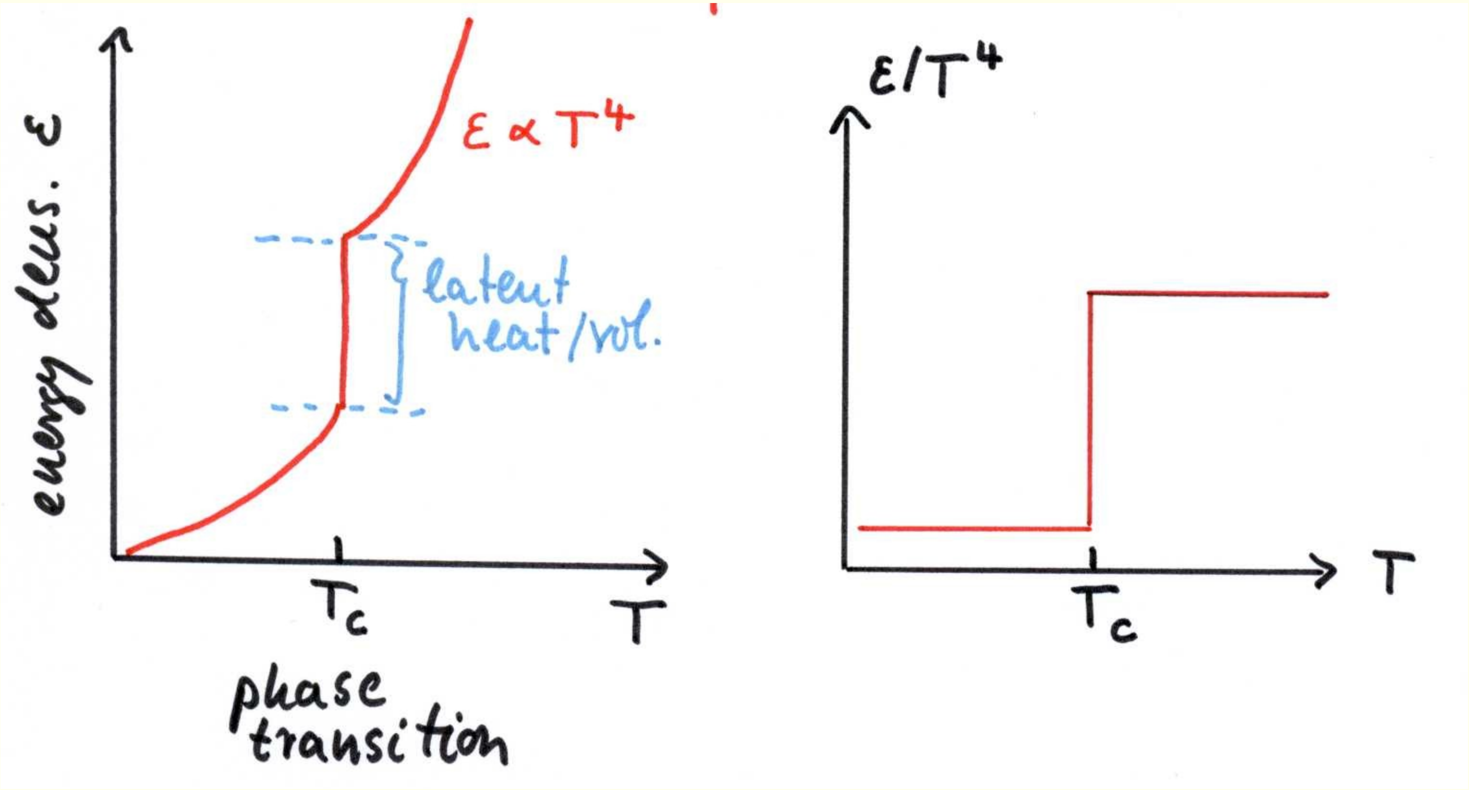


Suceptibilities χ: measure of fluctuations

$$\chi_L = N_\sigma^3(\langle L^2\rangle - \langle L\rangle^2) \qquad \chi_{\bar\psi\psi} = \partial\langle\bar\psi\psi\rangle/\partial m = \partial^2 p/\partial m^2$$

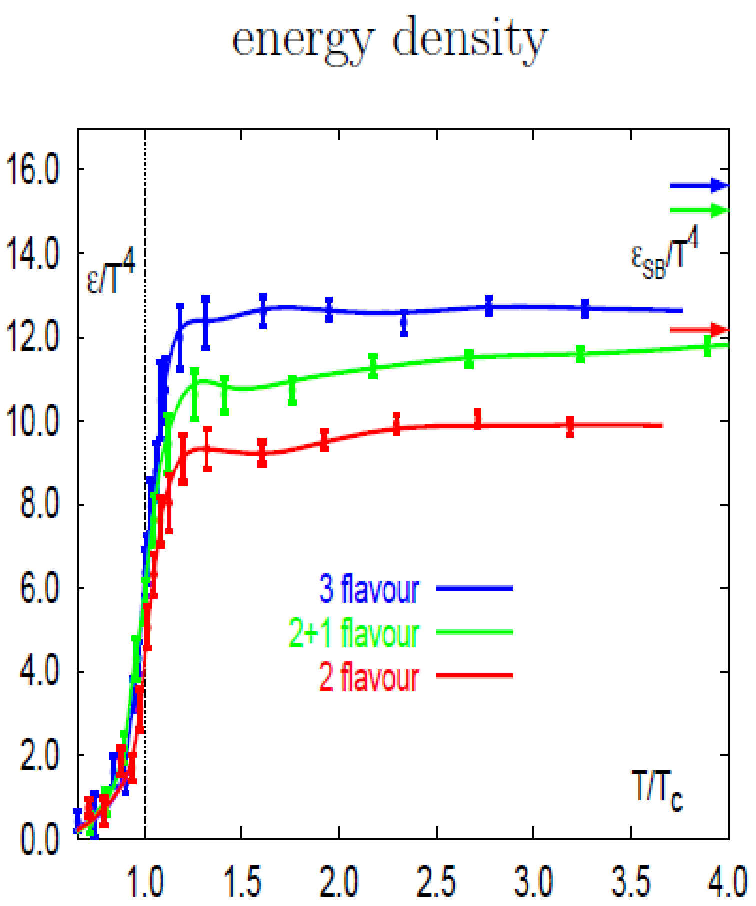
F. Karsch, E. Laermann, hep-lat/0305025

How to display equation of state?



divide by T^4 dependence for relativistic Bose/Fermi-gas

Equation of state in lattice QCD

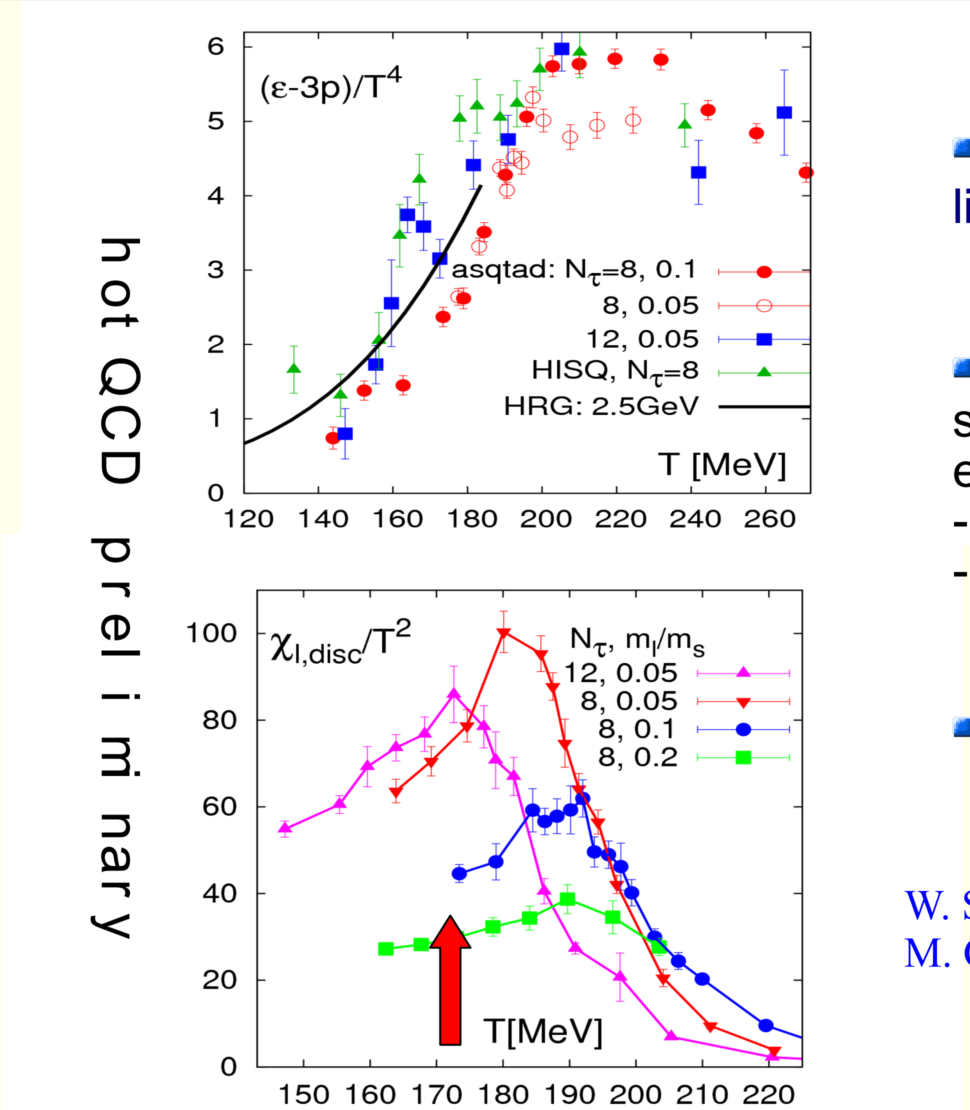


$\epsilon_c = 0.7\ \mathrm{GeV/fm^3}$

F. Karsch et al. Bielefeld group, Phys. Lett. B478(2000)447 and Nucl. Phys. A698(2002)199c

$16^3 \times 4$ lattice, $m_{ql}/T=0.4$, $m_{qh}/T=1$

What is most realistic value of the critical temperature?



- studies of the EoS with physical light and strange quark masses

  m_π = 220MeV and m_π =150MeV

- towards the continuum & chiral limits: systematic study of discretization errors and quark mass dependence
  - steady improvement, still not converged
  - need for computing power

- new continuum extrapolation of T_c

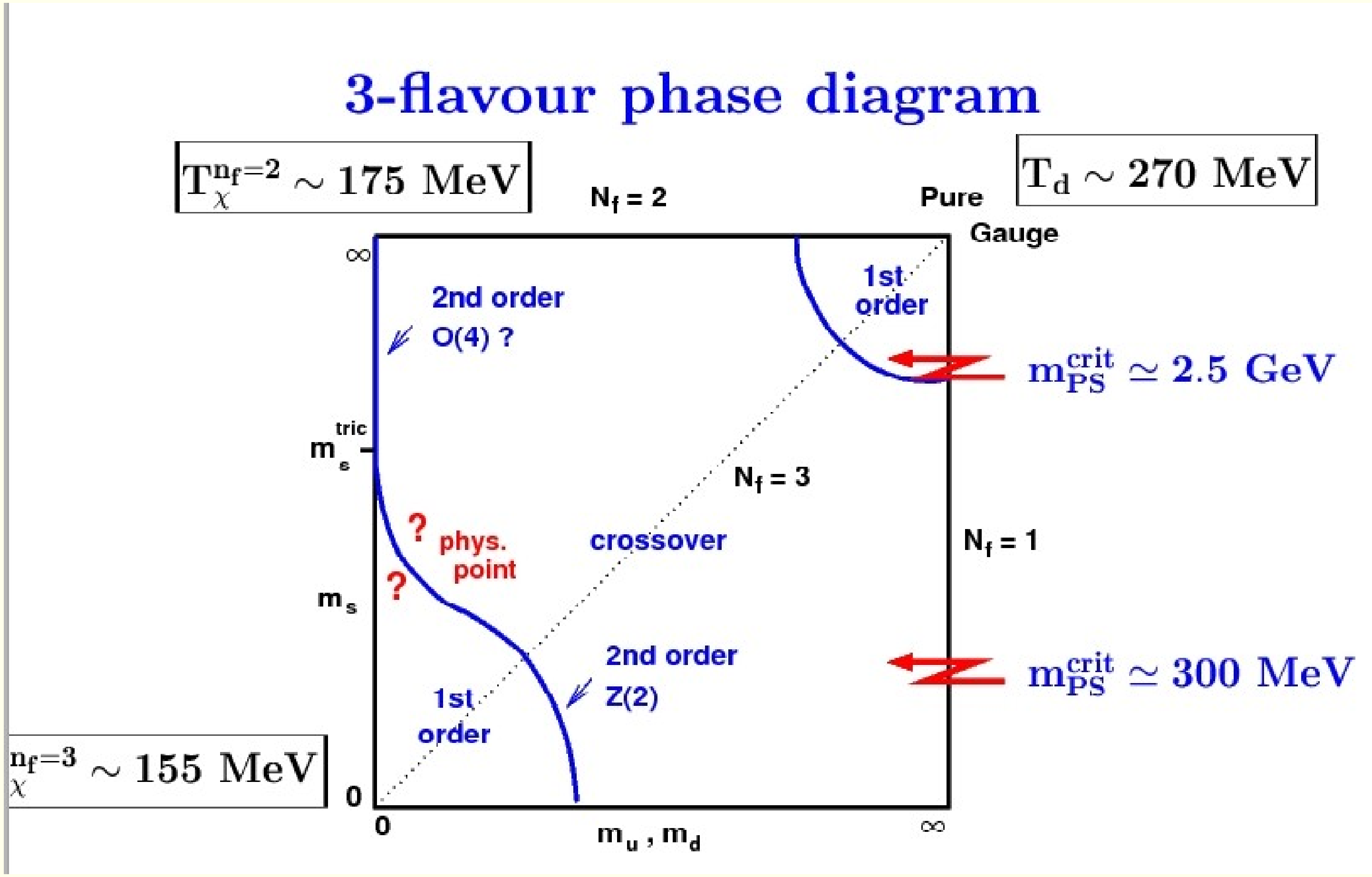
W. Soeldner, PoS Lattice2010 (2010) 215;
M. Cheng et.al., Phys. Rev. D81, 054504 (2010)

T_c from peak in chiral susceptibility

Order of phase transition

present state-of-the art lattice QCD simulations give smooth cross over for realistic quark masses

critical role of strange quark mass



Speed of sound from lattice QCD

similar to what was visible already for hadron gas – QGP using bag model equation of state:

in region of phase transition P/ε not constant

softest point and minimum in speed of sound