QGP Physics − from Fixed Target to LHC

2. Kinematic Variables

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Notations and Conventions

Natural units: $c = \hbar = 1$ also: $k_B = 1 \rightarrow E = k_B T,\ T = 2 \cdot 10^{12}\mathrm{K} \approx 172\,\mathrm{MeV}$

useful: $1 = \hbar c = 0.197\,\mathrm{GeV} \cdot \mathrm{fm}$

Space-time coordinates (contravariant vector): $x^\nu = (x^0, x^1, x^2, x^3) = (t, \vec{x}) = (t, x, y, z)$

Relativistic energy and momentum:

$$E = \gamma m, \quad p = \gamma \beta m, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad m = \text{rest mass}$$

4-momentum vector: $p^\mu = (p^0, p^1, p^2, p^3) = (E, \vec{p}) = (E, \vec{p}_T, p_z) = (E, p_x, p_y, p_z)$

Scalar product of two 4-vectors *a* and *b*: $a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$

Relation between energy and momentum: $E^2 = p^2 + m^2$

Center-of-Momentum System (CMS)

Consider a collision of two particles. The CMS is defined by $\vec{p}_a = -\vec{p}_b$

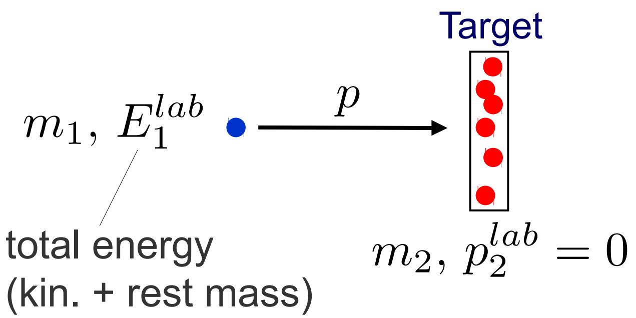
$$p_a = (E_a, \vec{p}_a) \qquad\qquad p_b = (E_b, \vec{p}_b)$$

The Mandelstam variable *s* is defined as $s := (p_a + p_b)^2 \overset{CMS}{=} (E_a + E_b)^2$

The center-of-mass energy √s is the total energy available in the CMS

√s for Fixed-Target und Collider Experiments (I)

Fixed-target experiment:



$$
\begin{aligned}
s &= \left[\begin{pmatrix} E_1^{\mathrm{lab}} \\ \vec{p_1} \end{pmatrix} + \begin{pmatrix} m_2 \\ \vec{0} \end{pmatrix} \right]^2 \\
&= {E_1^{\mathrm{lab}}}^2 + 2E_1^{\mathrm{lab}} m_2 + m_2^2 - p_1^2 \\
&= m_1^2 + m_2^2 + 2E_1^{\mathrm{lab}} m_2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \sqrt{s} &= \sqrt{m_1^2 + m_2^2 + 2E_1^{\mathrm{lab}} m_2} \\
&\stackrel{E_1^{\mathrm{lab}} \gg m_1, m_2}{\approx} \sqrt{2E_1^{\mathrm{lab}} m_2}
\end{aligned}
$$

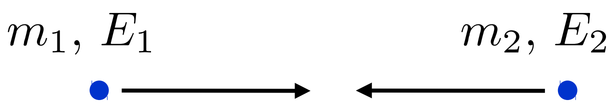
Example: Anti proton production
(fixed-target experiment): $p + p \to p + p + p + \bar{p}$

Minimum energy required to produce an anti-proton:
In CMS, all particles at rest after the reaction, i.e., √*s* = 4 m_p, hence:

$$
4m_p \stackrel{!}{=} \sqrt{2m_p^2 + 2E_1^{\mathrm{lab,min}} m_p} \quad \Rightarrow \quad E_1^{\mathrm{lab,min}} = \frac{(4m_p)^2 - 2m_p^2}{2m_p} = 7m_p
$$

√s for Fixed-Target und Collider Experiments (II)

Collider:



$$s = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$
$$= m_1^2 + m_2^2 + 2E_1 E_2 - 2\vec{p}_1\vec{p}_2$$

for $\vec{p}_1 = -\vec{p}_2$ and $m_1 = m_2$: $\sqrt{s} = 2E \quad \text{where} \;\; E \equiv E_1 = E_2$

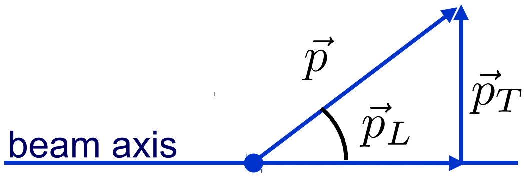
For heavy-ion collisions one usually states the center-of-momentum energy per Nucleon-nucleon pair:

Beam energy per nucleon (LHC Pb beam in 2010/11): $E = 1.38\,\text{TeV} \quad \rightarrow \quad \sqrt{s_{NN}} = 2.76\,\text{TeV}$

Rapidity

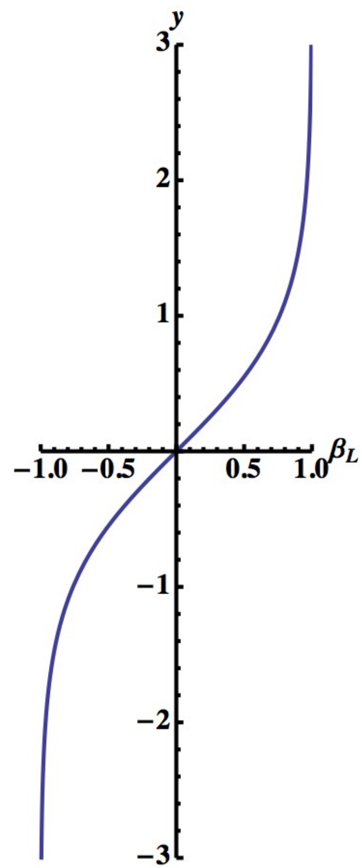
The rapidity y is a generalization of velocity $\beta_L = p_L / E$:

$$p = \sqrt{p_L^2 + p_T^2}$$



$$y := \operatorname{arctanh} \beta_L = \frac{1}{2} \ln \frac{1+\beta_L}{1-\beta_L} = \frac{1}{2} \ln \frac{E+p_L}{E-p_L}$$

For small velocities: $y \approx \beta_L$ for $\beta_L \ll 1$



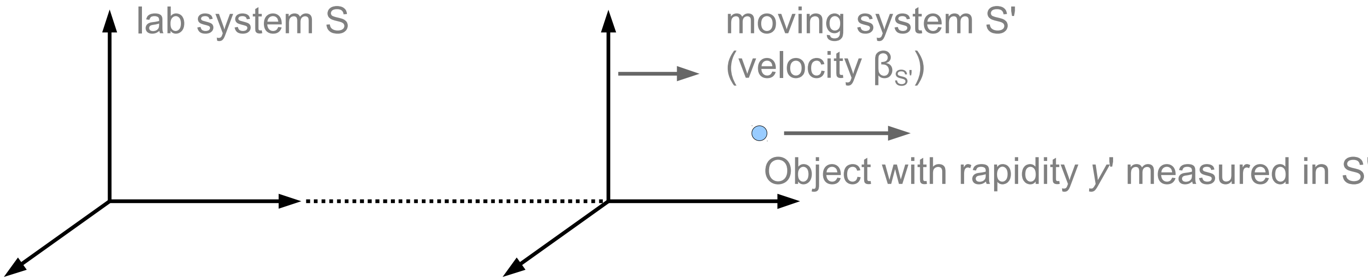
With $e^y = \sqrt{\dfrac{E+p_L}{E-p_L}}, \qquad e^{-y} = \sqrt{\dfrac{E-p_L}{E+p_L}}$

and $\sinh x = \dfrac{1}{2}\left(e^x - e^{-x}\right), \quad \cosh x = \dfrac{1}{2}\left(e^x + e^{-x}\right)$

we readily obtain $E = m_T \cdot \cosh y, \quad p_L = m_T \cdot \sinh y$

where $m_T := \sqrt{m^2 + p_T^2}$ is called the *transverse mass*

Additivity of Rapidity under Lorentz Transformation



Lorentz transformation: $E = \gamma(E' + \beta p'_z), \qquad p_z = \gamma(p'_z + \beta E') \qquad (\beta \equiv \beta_{S'})$

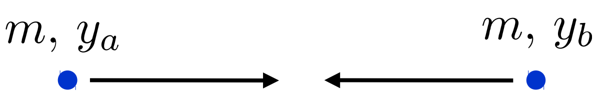
$$\begin{aligned} y &= \frac{1}{2}\ln\frac{E+p_z}{E-p_z} \\ &= \frac{1}{2}\ln\frac{\gamma(E'+\beta p'_z)+\gamma(p'_z+\beta E')}{\gamma(E'+\beta p'_z)-\gamma(p'_z+\beta E')} \\ &= \frac{1}{2}\ln\frac{(1+\beta)(E'+p'_z)}{(1-\beta)(E'-p'_z)} \\ &= \underbrace{\frac{1}{2}\ln\frac{1+\beta}{1-\beta}}_{\text{rapidity of S' as measured in S}} + \underbrace{\frac{1}{2}\ln\frac{E'+p'_z}{E'-p'_z}}_{y'} \end{aligned}$$

y is not Lorentz invariant, however, it has a simple transformation property:

$$y = y' + y_{S'}$$

Rapidity of the CMS

Consider collisions of two particles with equal mass m and rapidities y_a and y_b.
The rapidity of the CMS y_{CM} is then given by:



$$y_{\mathrm{CM}} = (y_a + y_b)/2$$

In the center-of-mass frame, the rapidities of a and b are:

$$y_a^* = -(y_b - y_a)/2 \quad \text{and} \quad y_b^* = (y_b - y_a)/2$$

Examples:

a) fixed target experiment: $y_{CM} = (y_{\mathrm{target}} + y_{\mathrm{beam}})/2 = y_{\mathrm{beam}}/2$

b) Collider: $y_{CM} = (y_{\mathrm{target}} + y_{\mathrm{beam}})/2 = 0$

Pseudorapidity

$$y = \frac{1}{2} \ln \frac{E + p\cos\vartheta}{E - p\cos\vartheta} \stackrel{p \gg m}{\approx} \frac{1}{2} \ln \frac{1+\cos\vartheta}{1-\cos\vartheta} = \frac{1}{2} \ln \frac{2\cos^2\frac{\vartheta}{2}}{2\sin^2\frac{\vartheta}{2}} = -\ln\left[\tan\frac{\vartheta}{2}\right] =: \eta$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

Special case: $y = \eta \quad \text{for} \quad m = 0$

Analogous to the relations for the rapidity we find

$$p = p_T \cdot \cosh\eta, \qquad p_L = p_T \cdot \sinh\eta$$

Example: Beam Rapidities

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \ln \frac{E + p_z}{\sqrt{E^2 - p_z^2}} = \ln \frac{E + p_z}{m} \approx \ln \frac{2E}{m}$$

Beam momentum (GeV/*c*)	Beam rapidity
100	5.36
158	5.81
1380	7.99
2750	8.86
3500	8.92
7000	9.61

Quick Overview: Kinematic Variables

Transverse momentum

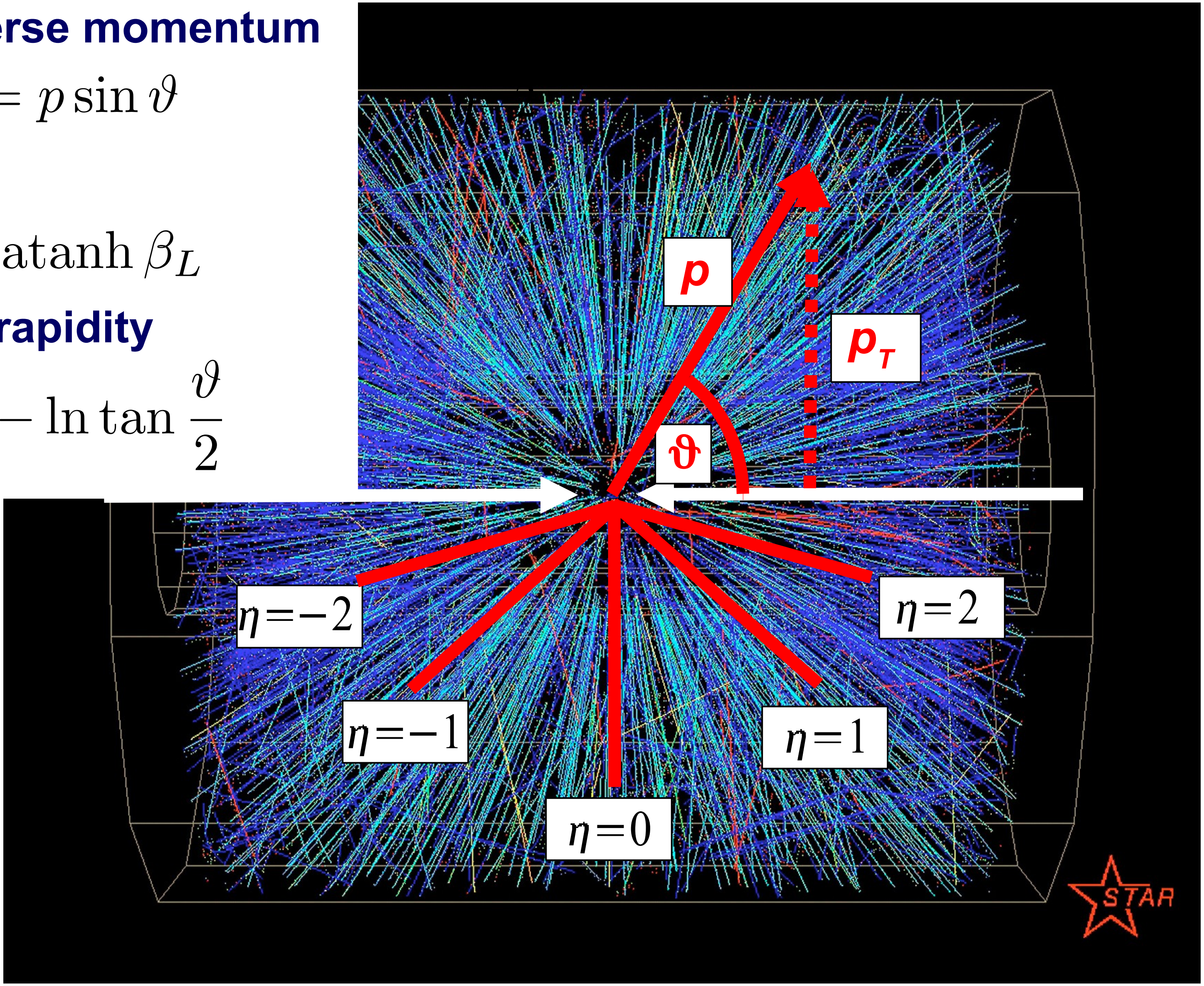
$$p_T = p \sin \vartheta$$

rapidity

$$y = \operatorname{atanh} \beta_L$$

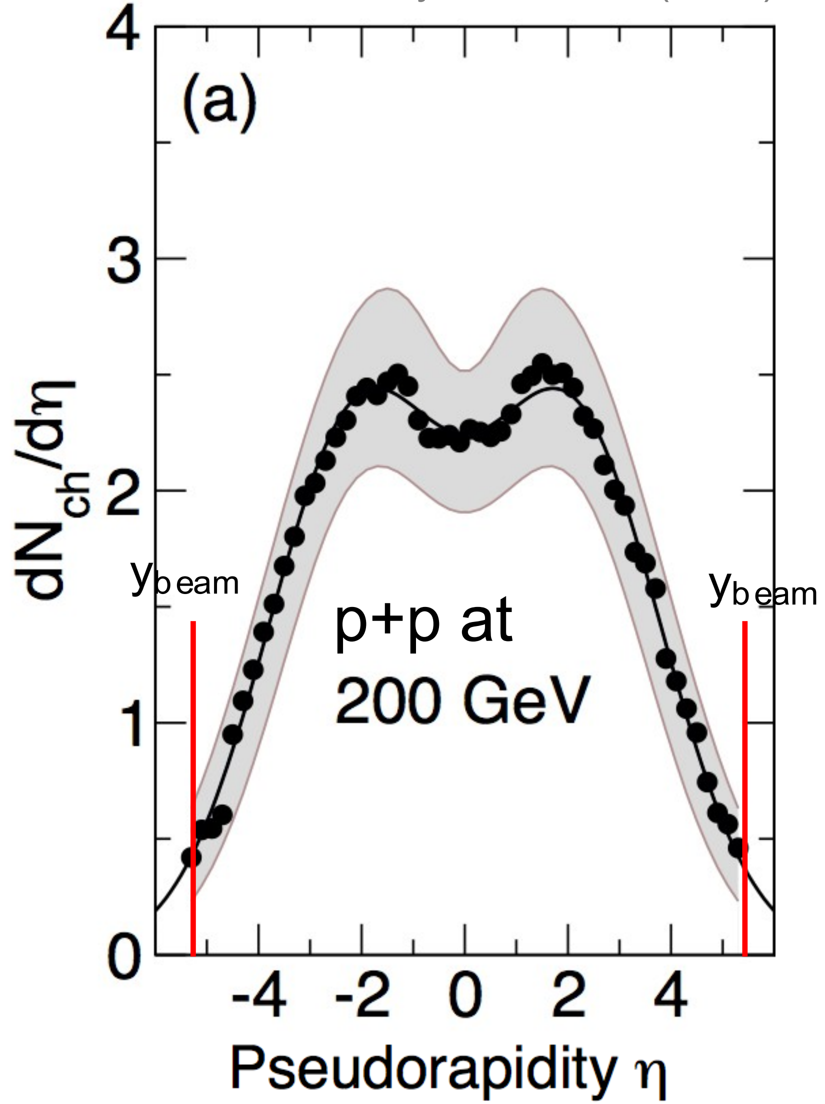
pseudorapidity

$$\eta = -\ln \tan \frac{\vartheta}{2}$$



Example of a Pseudo-rapidity Distribution

PHOBOS, Phys.Rev. C83 (2011) 024913



Beam rapidity:

$$y_{beam} = \ln \frac{E+p}{m} = 5.36$$

Average number of charged particles per collision:

$$\langle N_{ch} \rangle = \int \frac{dN_{ch}}{d\eta} d\eta \approx 20$$

Difference between dN/dy and dN/dη in the CMS

$$\frac{dN}{d\eta} = \sqrt{1 - \frac{m^2}{m_T^2 \cosh^2 y}} \frac{dN}{dy}$$

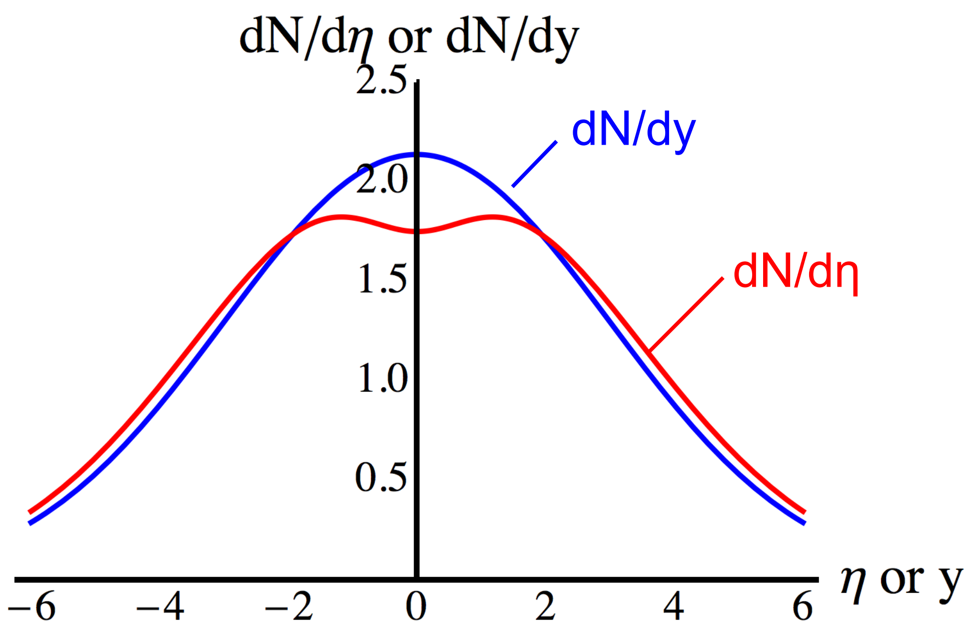
$$y(\eta) = \frac{1}{2} \log \left(\frac{\sqrt{p_T^2 \cosh^2(\eta) + m^2} + p_T \sinh(\eta)}{\sqrt{p_T^2 \cosh^2(\eta) + m^2} - p_T \sinh(\eta)} \right)$$

Difference between dN/dy and dN/dη in the CMS at y = 0:

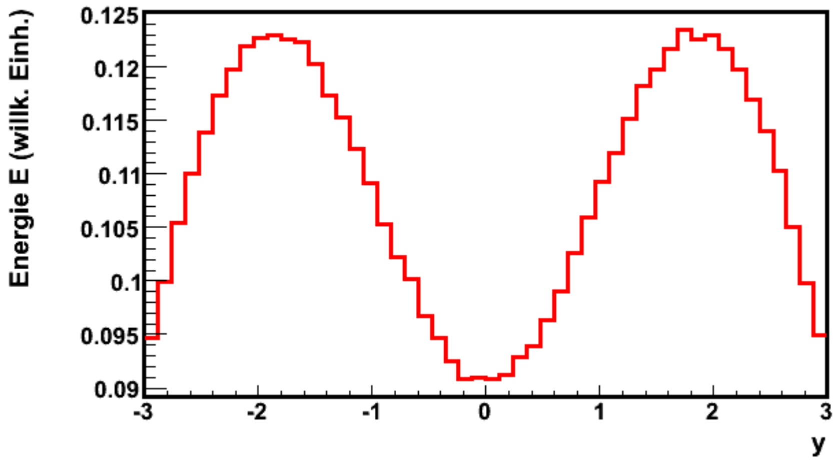
Simple example:
Pions distributed according to

$$\frac{1}{2\pi p_T} \frac{d^2N}{dp_T dy} = G(y) \cdot \exp(-p_T/0.16)$$

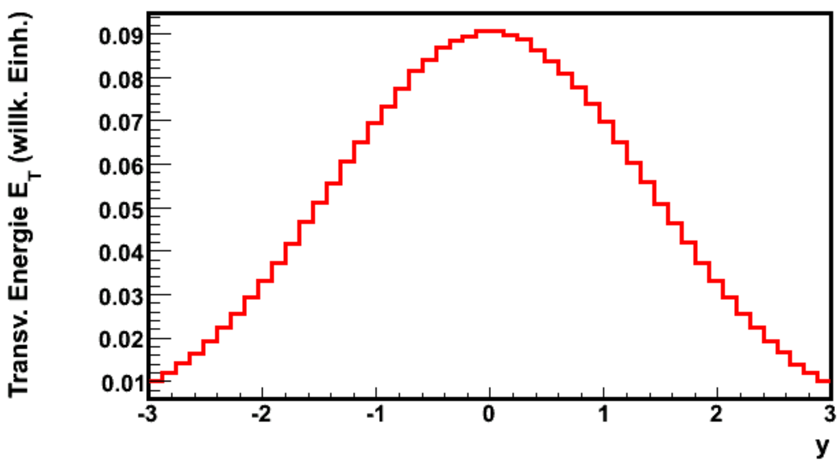
Gaussian with σ=3



Total Energy and Transverse Energy



$$\frac{dE}{dy} \qquad E = \sqrt{m^2 + p^2}$$



$$\frac{dm_T}{dy} \qquad m_T = \sqrt{m^2 + p_T^2}$$

Luminosity and Cross Sections (I)

The luminosity *L* of a collider is defined by:

$$\frac{\mathrm{d}N_\mathrm{int}}{\mathrm{d}t} = \sigma \cdot L$$

$$\begin{aligned} L &= \text{luminosity (in } \mathrm{s}^{-1}\mathrm{cm}^{-2}) \\ \mathrm{d}N_\mathrm{int}/\mathrm{d}t &= \text{Number of interactions of a certain type per second} \\ \sigma &= \text{cross section for this reaction} \end{aligned}$$

$$L = \frac{n_1 n_2 f}{A}$$

$$\begin{aligned} n_1,\, n_2 &= \text{numbers of particles per bunch in the two beams} \\ f &= \text{bunch crossing frequency at a given crossing point} \\ A &= \text{beam crossing area} \end{aligned}$$

Luminosity and Cross Sections (II)

The luminosity can be determined by measuring the beam current:

$$I_{1,2} = n_{1,2} \cdot N_b \cdot e \cdot f$$

$$\begin{aligned} N_b &= \text{number of bunches in the beam} \\ e &= \text{elementary electric charge} \end{aligned}$$

The crossing area *A* is usually calculated as

$$A = 4\pi\sigma_x\sigma_y$$

The standard deviations of the beam profiles are measured by sweeping the beams transversely across each other in a so called van der Meer scan.

Integrated luminosity:

$$L_{\text{int}} = \int L \, \mathrm{d}t$$

Lorentz invariant Phase Space Element

If one is interested in the production of a particle *A* one could define the observable

$$\frac{1}{L_{\text{int}}} \frac{d^3 N_A}{d^3 \vec{p}} = \frac{1}{L_{\text{int}}} \frac{d^3 N_A}{dp_x dp_y dp_z}$$

However, the phase space density would then not be Lorentz invariant:

$$\frac{d^3 N}{dp'_x dp'_y dp'_z} = \frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} \cdot \frac{d^3 N}{dp_x dp_y dp_z} = \frac{E}{E'} \cdot \frac{d^3 N}{dp_x dp_y dp_z}$$

We thus use the Lorentz invariant phase space element $\frac{d^3 \vec{p}}{E} = \frac{dp_x dp_y dp_z}{E}$

The corresponding observable is called Lorentz invariant cross section:

$$E \frac{d^3 \sigma}{d^3 \vec{p}} = \frac{1}{L_{\text{int}}} E \frac{d^3 N}{d^3 \vec{p}} = \underbrace{\frac{1}{N_{\text{evt,tot}}} E \frac{d^3 N}{d^3 \vec{p}}}_{\text{this is called the invariant yield}} \sigma_{\text{tot}}$$

[Lorentz invariant Phase Space Element: Proof of Invariance]

Lorentz boost along the z axis:

$$\begin{aligned} p'_x &= p_x \\ p'_y &= p_y \\ p'_z &= \gamma(p_z - \beta E), \qquad p_z = \gamma(p'_z + \beta E') \\ E' &= \gamma(E - \beta p_z), \qquad E = \gamma(E' + \beta p'_z) \end{aligned}$$

Jacobian:

$$\frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} = \begin{vmatrix} \frac{\partial p_x}{\partial p'_x} & 0 & 0 \\ 0 & \frac{\partial p_y}{\partial p'_y} & 0 \\ 0 & 0 & \frac{\partial p_z}{\partial p'_z} \end{vmatrix}$$

$$\frac{\partial p_x}{\partial p'_x} = 1, \quad \frac{\partial p_y}{\partial p'_y} = 1, \quad \frac{\partial p_z}{\partial p'_z} = \frac{\partial}{\partial p'_z}\left[\gamma\left(p'_z + \beta E'\right)\right] = \gamma\left(1 + \beta \frac{\partial E'}{\partial p'_z}\right)$$

$$\frac{\partial E'}{\partial p'_z} = \frac{\partial}{\partial p'_z}\left[\left(m^2 + p'^2_x + p'^2_y + p'^2_z\right)^{1/2}\right] = \frac{p'_z}{E'}$$

$$\rightsquigarrow \frac{\partial p_z}{\partial p'_z} = \gamma\left(1 + \beta \frac{p'_z}{E'}\right) = \frac{E}{E'}$$

And so we finally obtain:

$$\frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} = \frac{E}{E'}$$

Invariant Cross Section

Invariant cross section in practice:

$$E\frac{d^3\sigma}{d^3p} = E\frac{1}{p_T}\frac{d^3\sigma}{dp_T\,dp_z\,d\varphi} \overset{dp_z/dy = m_T\cosh y = E}{=} \frac{1}{p_T}\frac{d^3\sigma}{dp_T\,dy\,d\varphi} \overset{\text{symmetry in }\varphi}{=} \frac{1}{2\pi p_T}\frac{d^2\sigma}{dp_T\,dy}$$

Sometimes also measured as a function of m_T:

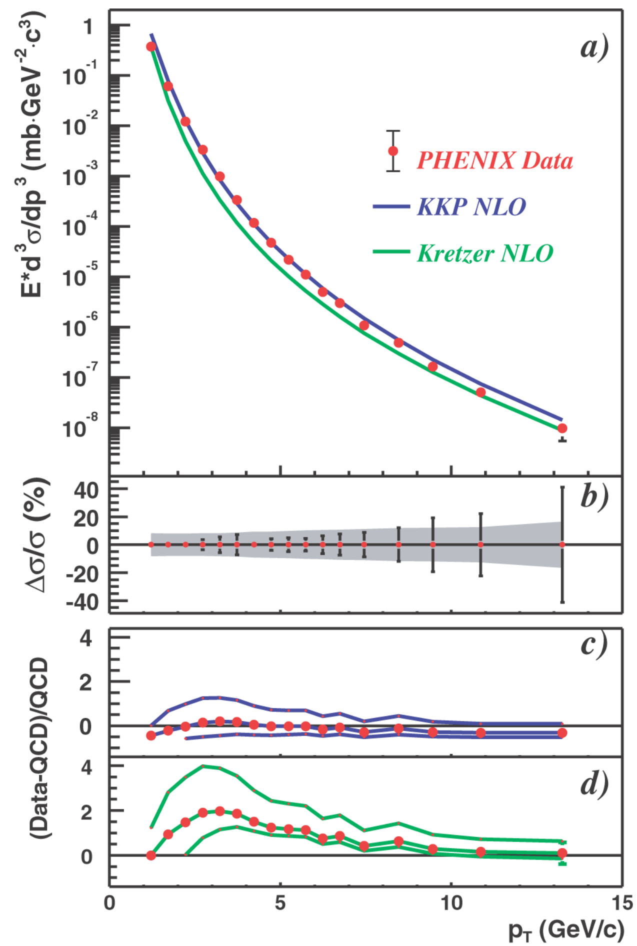
$$\frac{1}{2\pi m_T}\frac{d^2\sigma}{dm_T\,dy} = \frac{1}{2\pi m_T}\frac{d^2\sigma}{dp_T\,dy}\frac{dp_T}{dm_T} = \frac{1}{2\pi p_T}\frac{d^2\sigma}{dp_T\,dy}$$

Integral of the inv. cross section:

$$\int E\frac{d^3\sigma}{d^3p}\,d^3p/E = \langle N_x\rangle \cdot \sigma_{\text{tot}}$$

Average yield of particle *X* per event

Example: Invariant cross section for neutral pion production in p+p at √*s* = 200 GeV



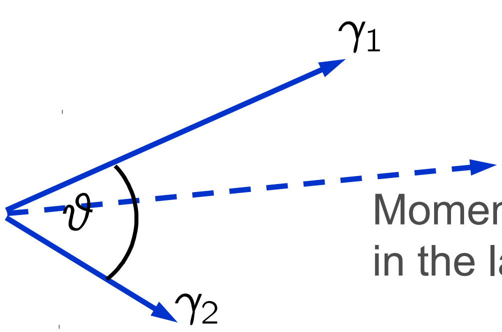
Invariant Mass

Consider the decay of a particle in two daughter particles. The mass of the mother particle is given by ("invariant mass"):

$$
\begin{aligned}
M^2 &= \left[\begin{pmatrix} E_1 \\ \vec{p}_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ \vec{p}_2 \end{pmatrix} \right]^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \\
&= m_1^2 + m_2^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2 \\
&= m_1^1 + m_2^2 + 2E_1E_2 - 2p_1p_2 \cos\vartheta
\end{aligned}
$$

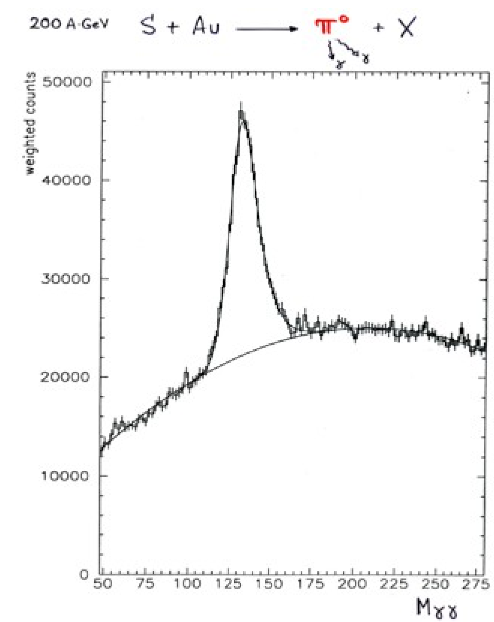
Example: π^0 decay: $\pi^0 \to \gamma + \gamma,\ m_1 = m_2 = 0,\ E_i = p_i$

$$\Rightarrow M = \sqrt{2E_1E_2(1 - \cos\vartheta)}$$



Momentum of the π^0 in the lab. system

Example: π^0 peak with combinatorial background



Points to Take Home

Center-of-mass energy $\sqrt{s}$: Total energy in the center-of-mass (or momentum) system (rest mass of + kinetic energy)

Observables: Transverse momentum p_T and rapidity y

Pseudorapidity $\eta \approx y$ for $E >> m$ ($\eta = y$ for $m = 0$, e.g., for photons)

Production rates of particles describes by the Lorentz invariant cross section:

Lorentz-invariant cross section: $$E \frac{d^3\sigma}{d^3p}$$