QGP Physics − From Fixed Target to LHC

4. Thermodynamics of the QGP

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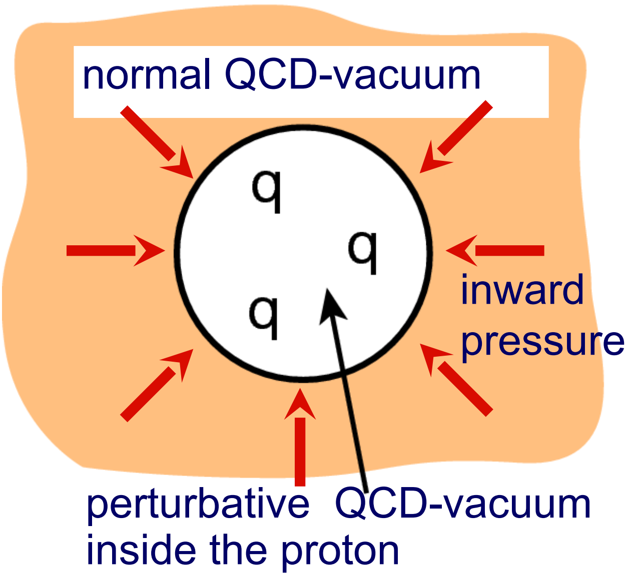
MIT Bag Model

A. Chodos et al., Phys. Rev. D10 (1974) 2599
T. DeGrand et al. Phys. Rev. D12 (1975) 2060



- Build confinement and asymptotic freedom into simple phenomenological model
- Hadron = „bag" filled with quarks
- Two kinds of vacuum
- Normal QCD-Vacuum outside of the bag
- Perturbative QCD-Vacuum within the bag

Energy density: $\varepsilon = E/V$

Energy density in the bag is higher than in the vacuum:

$$\varepsilon_{\text{pert}} - \varepsilon_{\text{vacuum}} =: B$$

Energy of *N* quarks in a bag of radius *R*:

$$E = \frac{2.04N}{R} + \frac{4}{3}\pi R^3 B$$

kinetic energy of *N* particles in a spherical box of radius *R*

Condition for stability: d*E*/d*R* = 0 (minimum):

$$B^{1/4} = \left(\frac{2.04N}{4\pi}\right)^{1/4} \frac{1}{R} \quad \overset{N=3,\, R=0.8\,\text{fm}}{\Longrightarrow} \quad B^{1/4} = 206\,\text{MeV} \quad (\hbar = c = 1)$$

Particles in a Box: Number of States

Number of states between momentum p and p+dp
(each state occupies a volume h^3 in phase space):

number of states

physical volume

$$dN = \frac{V}{h^3} \boxed{4\pi p^2 dp}$$

volume of a spherical shell with radius *p*
and thickness d*p* in momentum space

Fermi-Dirac and Bose-Einstein Distribution

Let‘s consider an ideal gas of bosons and fermions (grand canonical ensemble).

Average occupation number of a state is

$$n(E) = \frac{g}{\exp\left[\frac{E-\mu}{kT}\right] + 1}$$

... for fermions (half-integer spin):

(Fermi-Dirac distribution)

$$n(E) = \frac{g}{\exp\left[\frac{E-\mu}{kT}\right] - 1}$$

... for bosons (integer spin):

(Bose-Einstein distribution)

g : # degrees of freedom (degeneracy)

μ : chemical potential

T : temperature

Degeneracy

QGP:

$$g_{\text{bosons}} = 8_{\text{color}} \times 2_{\text{spin}} = 16$$

$$\begin{aligned} g_{\text{fermions}} &= g_{\text{quark}} + g_{\text{anti-quark}} = 2 \times g_{\text{quark}} \\ &= 2 \times 2_{\text{spin}} \times 2_{\text{flavor}} \times 3_{\text{color}} = 24 \end{aligned}$$

assume only u and d quarks can be produced in the QGP, the rest is too heavy

Pion gas:

$$g_{\text{bosons}} = 3_{\text{type}} \qquad g_{\text{fermions}} = 0$$

(π^+, π^-, π^0)

Total Quark Density in the Ideal (= non interacting) QGP at Temperature *T*

Quark density (Fermi-Dirac distribution, massless quarks, i.e., $E = p$):

$$n_q(\mu_q) = \frac{N_q}{V} = g_q \frac{4\pi}{(2\pi)^3} \int_0^\infty dE \frac{E^2}{e^{(E-\mu_q)/T} + 1} \qquad (\hbar = k = 1)$$

For massless quarks: $\mu_q + \mu_{\bar{q}} = 0$

Thus, we obtain for antiquarks:

$$n_{\bar{q}}(\mu_q) = g_{\bar{q}} \frac{4\pi}{(2\pi)^3} \int_0^\infty dE \frac{E^2}{e^{(E+\mu_q)/T} + 1} \qquad (\hbar = k = 1)$$

Quark-Gluon Plasma with μ = 0: Quarks

1.20205

Quark density ($\mu_q = 0$):

$$\boxed{n_q = n_{\bar{q}} = \frac{g_q}{2\pi^2}\frac{3}{2}\zeta(3)T^3}$$

Total energy of the quarks (*E* = *p* for massless quarks):

$$E = \int_0^\infty E\, dN_q$$

Energy density and pressure ($\mu_q = 0$):

$$\boxed{\varepsilon_q = \frac{E_q}{V} = \frac{7}{8} g_q \frac{\pi^2}{30} T^4, \qquad p_q = \frac{1}{3}\varepsilon_q}$$

(identical result for antiquarks ($\mu_q = 0$))

Example: $T = 200\,\mathrm{MeV},\ g_q = 18 \quad \Rightarrow \quad n_q = n_{\bar{q}} = 1.71/\mathrm{fm}^3$

Quark-Gluon Plasma: Gluons

Gluons (μ = 0), Bose-Einstein distribution:

$$n_g = g_g \frac{4\pi}{(2\pi)^3} \int_0^\infty dE \frac{E^2}{e^{E/T} - 1}, \qquad \varepsilon_g = g_g \frac{4\pi}{(2\pi)^3} \int_0^\infty dE \frac{E^3}{e^{E/T} - 1} \qquad (\hbar = k = 1)$$

Solution:

Energy density:	Pressure:	Particle density:
$\varepsilon_g = g_g \frac{\pi^2}{30} T^4,$	$p_g = \frac{1}{3} \varepsilon_g,$	$n_g = \frac{g_g}{\pi^2} \zeta(3) T^3$

Example: $T = 200\,\mathrm{MeV},\ g_q = 16 \quad \Rightarrow \quad n_g = 2.03\ \mathrm{gluons/fm^3}$

Summary: Quark-Gluon Plasma with μ = 0: Pressure and Energy Density

Pressure and energy density in a quark-gluon plasma at μ = 0 without particle interactions:

$$p_{\mathrm{QGP}} = \left(g_g + \frac{7}{8}(g_q + g_{\bar{q}})\right)\frac{\pi^2}{90}T^4, \qquad \varepsilon_{\mathrm{QGP}} = 3p_{\mathrm{QGP}}$$

$$= \begin{cases} 37\dfrac{\pi^2}{90}T^4 & \text{for } u, d \\ 47.5\dfrac{\pi^2}{90}T^4 & \text{for } u, d, s \end{cases} \qquad = \begin{cases} 37\dfrac{\pi^2}{30}T^4 & \text{for } u, d \\ 47.5\dfrac{\pi^2}{30}T^4 & \text{for } u, d, s \end{cases}$$

Example:

$$T = 200\,\mathrm{MeV},\ \text{two active quark flavors} \quad \Rightarrow \quad \varepsilon_{\mathrm{QGP}}^{\mathrm{id.\,gad}} = 2.55\ \mathrm{GeV/fm^3}$$

Quark-Gluon Plasma with μ = 0: Critical Temperature (I)

Accounting for the QCD-vacuum:

$$\varepsilon_{\mathrm{QGP}}^{\mathrm{QCD\ vac.}} = \varepsilon_{\mathrm{QGP}} + B$$

$$p_{\mathrm{QGP}}^{\mathrm{QCD\ vac.}} = p_{\mathrm{QGP}} - B$$

$$E = TS - pV \quad (\mu = 0)$$

$$\Rightarrow p = Ts - \varepsilon$$

So we have:

$$p_{\mathrm{HG}} = 3aT^4 \qquad \varepsilon_{\mathrm{HG}} = 9aT^4$$

$$p_{\mathrm{QGP}}^{\mathrm{QCD\ vac.}} = 37aT^4 - B \qquad \varepsilon_{\mathrm{QGP}}^{\mathrm{QCD\ vac.}} = 111aT^4 + B$$

$$a = \frac{\pi^2}{90}$$

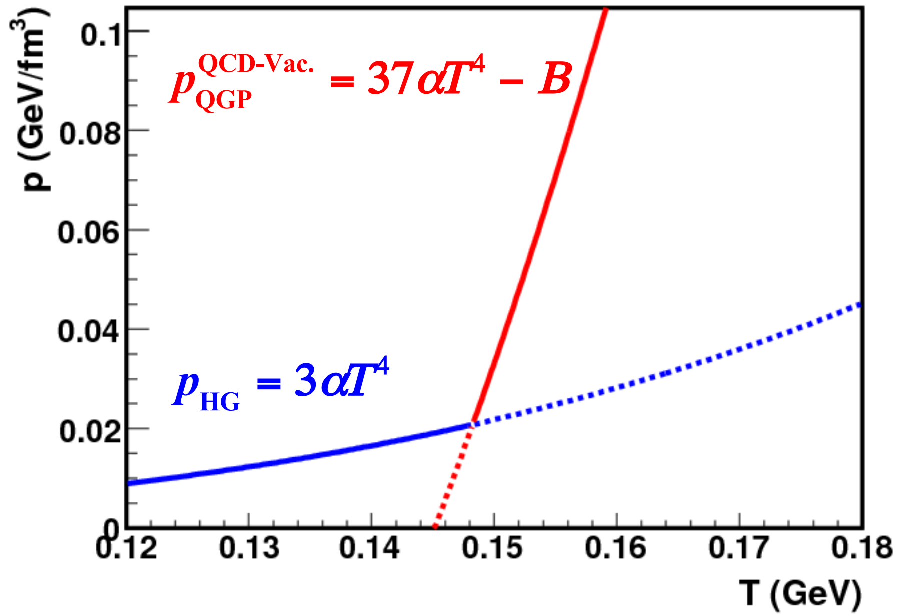
Gibbs criterion for the phase transition:

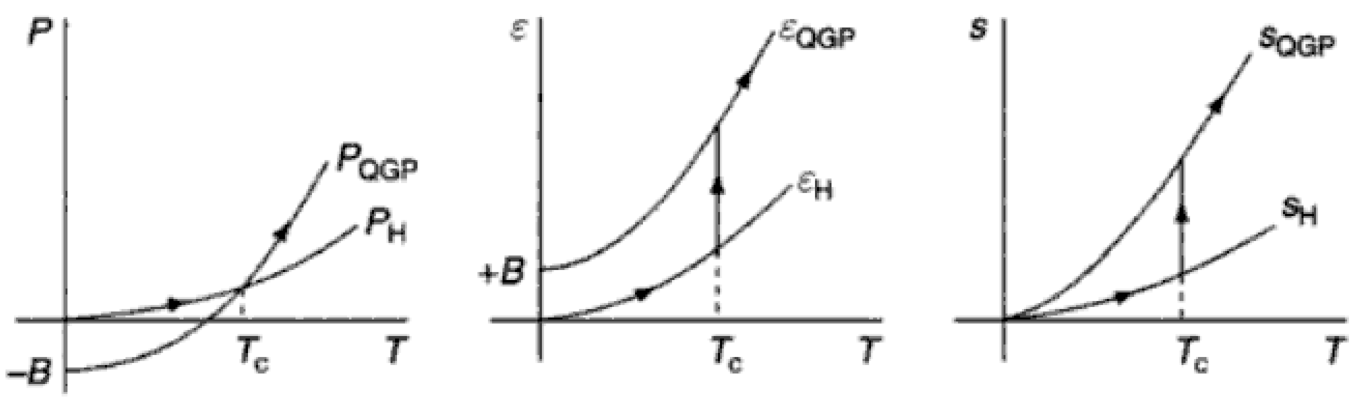
$$p_{\mathrm{HG}}(T_c) = p_{\mathrm{QGP}}^{\mathrm{QCD\ vac.}}(T_c) \quad \Rightarrow \quad T_c = \left(\frac{B}{34a}\right)^{1/4} \approx 150\,\mathrm{MeV}$$

Phase transition in the bag model is of first order. Latent heat:

$$\varepsilon_{\mathrm{QGP}}^{\mathrm{QCD\ vac.}}(T_c) - \varepsilon_{\mathrm{HG}}(T_c) = 102aT_c^4 + B = 4B$$

Quark-Gluon Plasma with μ = 0: Critical Temperature (II)





Quark-Gluon Plasma with μ = 0: Entropy

Entropy density for constant temperature and pressure:

$$\varepsilon = Ts + \mu N - p \quad \stackrel{\mu=0}{\Rightarrow} \quad s = \frac{\varepsilon + p}{T} = 4\frac{p}{T}$$

Ratio of entropy density (QGP / pion gas):

$$s_{\mathrm{QGP}} = 148aT^3, \quad s_{\mathrm{HG}} = 12aT^3, \quad \Rightarrow \quad \frac{s_{\mathrm{QGP}}}{s_{\mathrm{HG}}} \approx 12.3$$

Entropy per particle:

Pion gas: $$\frac{s_{\mathrm{HG}}}{n_\pi} = \frac{12\pi^2/90\,T^3}{g_\pi/\pi^2 1.202\,T^3} = 3.6$$

QGP: $$\frac{s_q}{n_q} = 1.4, \qquad \frac{s_g}{n_g} = 1.2$$

Quark-Gluon Plasma with $\mu \neq 0$: Energy and Particle Number Density of the Quarks

For $\mu_q \neq 0$ a solution in closed form can be found for $\varepsilon_q + \varepsilon_{\bar{q}}$ but not for ε_q and $\varepsilon_{\bar{q}}$ separately: Chin, PL 78B (1978) 552

$$\varepsilon_q + \varepsilon_{\bar{q}} = g_q \times \left(\frac{7\pi^2}{120} T^4 + \frac{\mu_q^2}{4} T^2 + \frac{\mu_q^4}{8\pi^2} \right)$$

Accordingly one finds for the quark density

$$n_q - n_{\bar{q}} = g_q \times \left(\frac{\mu_q}{6} T^2 + \frac{\mu_q^3}{6\pi^2} \right)$$

From this the net baryon density can be determined as (for $g_q = 12$):

$$n_B = \frac{n_q - n_{\bar{q}}}{3} = \frac{2\mu_q}{3} T^2 + \frac{2\mu_q^3}{3\pi^2} = \frac{2\mu_B}{9} T^2 + \frac{2\mu_B^3}{81\pi^2} \qquad (\mu_B = 3\mu_q)$$

Quark-Gluon Plasma with μ ≠ 0: Critical Temperature and Critical Quark Potential

Energy density in a QGP with μ ≠ 0 (without particle interactions):

$$\varepsilon_{\mathrm{QGP}} = \varepsilon_q + \varepsilon_{\bar{q}} + \varepsilon_g = \frac{37\pi^2}{30} T^4 + 3\mu_q^2 T^2 + \frac{3\mu_q^4}{2\pi^2}$$

Condition for QGP stability:

$$p_{\mathrm{QGP}} = \frac{1}{3}\varepsilon_{\mathrm{QGP}} \stackrel{!}{=} B \quad \Rightarrow T_c(\mu_q)$$

Condition for QGP: QGP-pressure ≥ pressure of the QCD-vacuum (similar, but not identical, to the previous condition $p_{\mathrm{HG}} = p_{\mathrm{QGP}}$)

Critical temperature / quark potential:

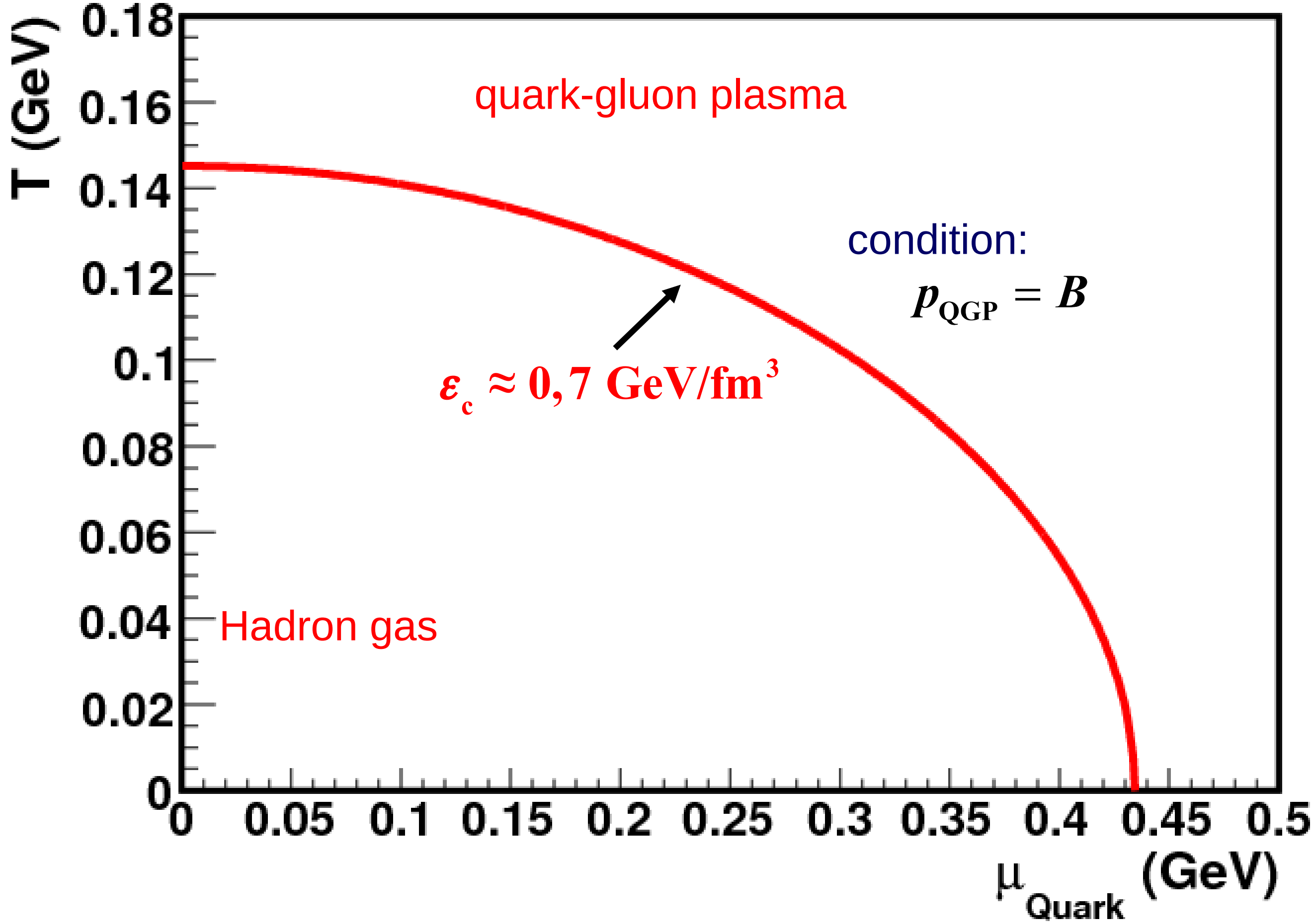
$$T_c(\mu_q = 0) = \left(\frac{90B}{37\pi^2}\right)^{1/4}$$

$$\mu_q^c(T=0) = (2\pi^2 B)^{1/4} = 0.43\,\mathrm{GeV}$$

$$n_B^c(T=0) = \frac{2}{3\pi^2}(2\pi^2 B)^{3/4} = 0.72\,\mathrm{fm}^{-3} \approx 5 \times n_{\mathrm{nucleus}}$$

Possibly reached in neutron stars

Quark-Gluon Plasma with $\mu \neq 0$: Phase Diagram of the Non-Interacting QGP



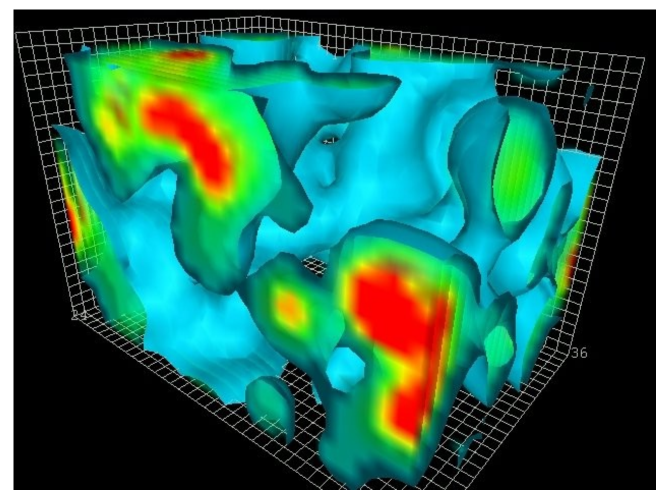
$\mu_B = 3 \cdot \mu_{Quark}$

Lattice QCD (I)

- QCD asymptotically free at extremely large *T* and/or small distances
- Cannot use perturbation theory to calculate, e.g., properties of hadrons
- Instead solve QCD numerically at zero and finite temperature by putting gauge fields on a space-time lattice ↔ “lattice QCD”
- First-principle non-perturbative calculation
- Lattice needs to be big, e.g. 16^3 x 32

Example of a machine for lattice QCD:
JUGENE in Jülich (294,912 processor cores, ~ 1 PetaFLOPS)

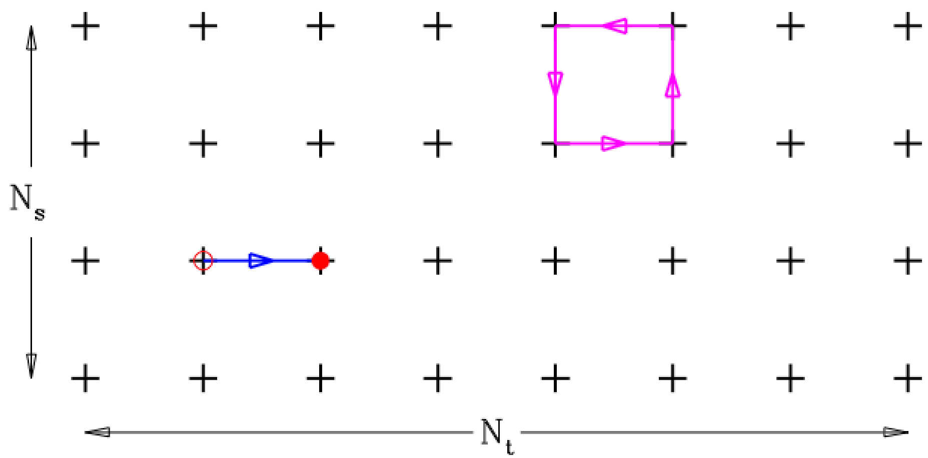
Snapshot of fluctuating quark and gluon flields on a discrete space-time lattice:



http://www.physics.adelaide.edu.au/theory/staff/leinweber/index.html

Lattice QCD (II)

Lattice spacing $a, \quad a^{-1} \sim \Lambda_{\mathrm{UV}}, \quad x_\mu = n_\mu a$
Finite volume $L^3 \cdot T, \quad N_s = L/a, \quad N_t = T/a$

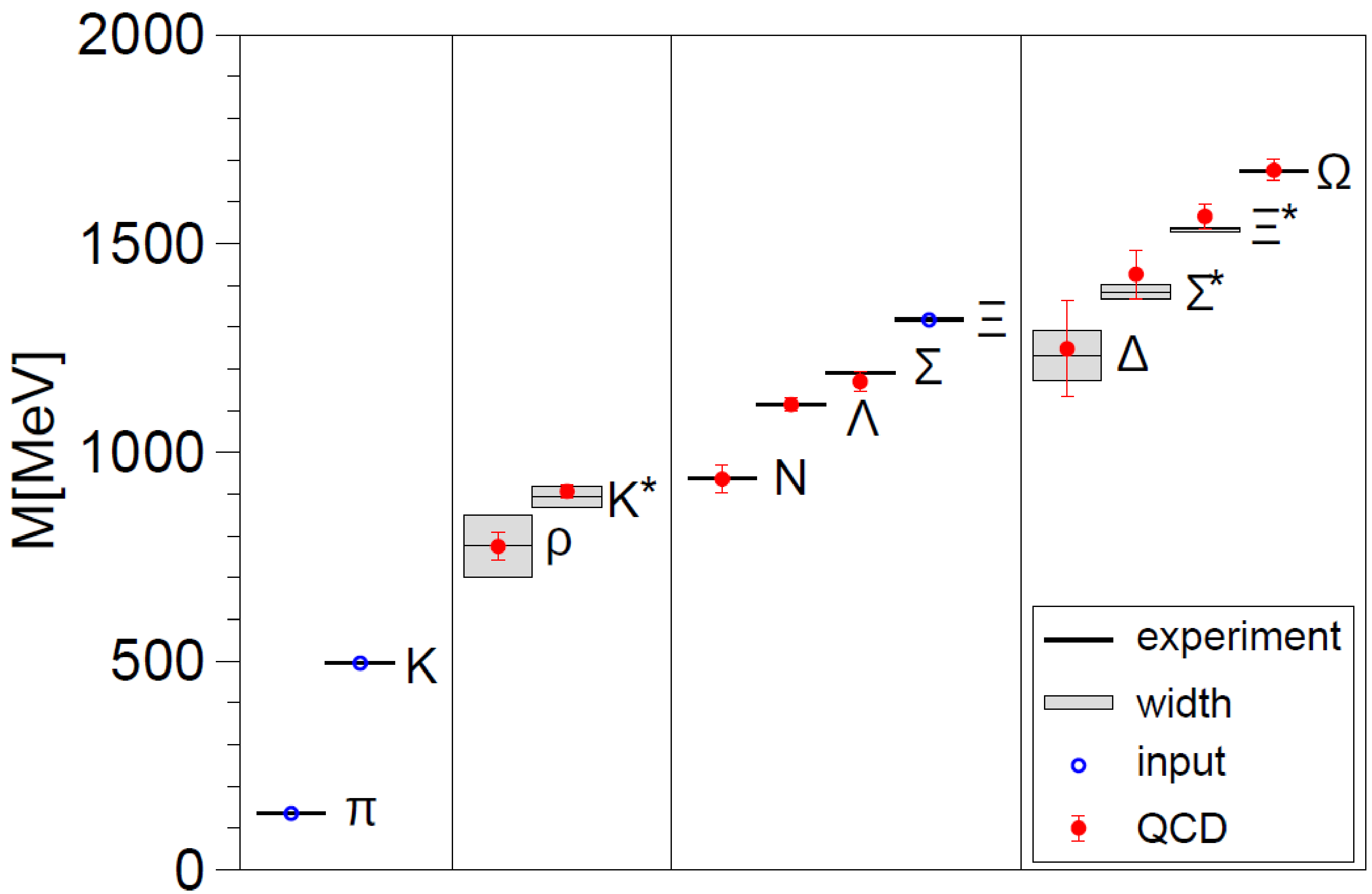


(anti)quarks:	$\psi(x), \overline{\psi}(x)$	lattice sites
gluons:	$U_\mu(x) = \mathrm{e}^{aA_\mu(x)} \in \mathrm{SU}(3)$	links
field tensor:	$P_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x)$	"plaquettes"

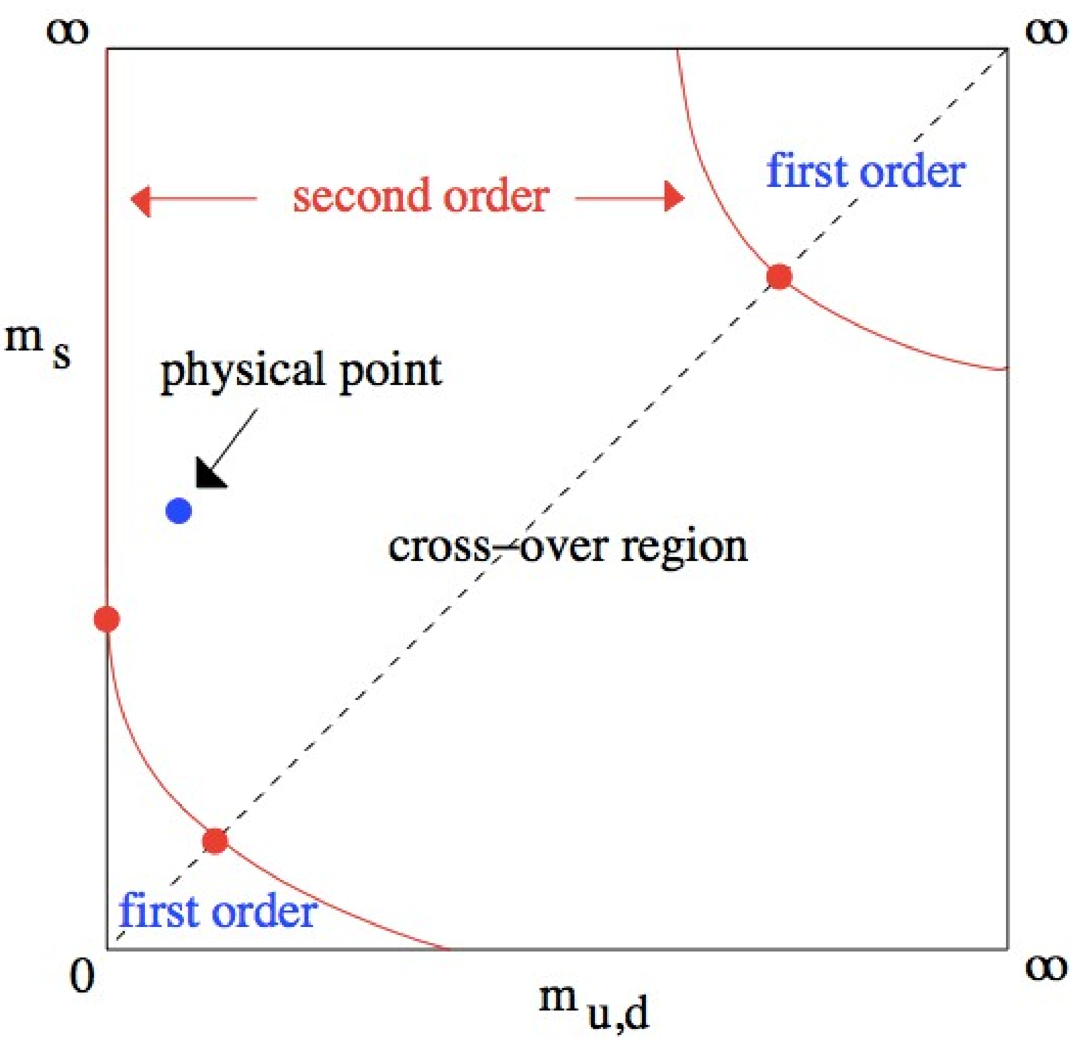
from Hartmut Wittig: Lattice QCD - Introduction and Results (→ link)

Hadron Spectrum from Lattice QCD

S. Dürr, Z.Fodor et al., Budapest-Marseille–Wuppertal Coll.,Science 322 (2008) 1225



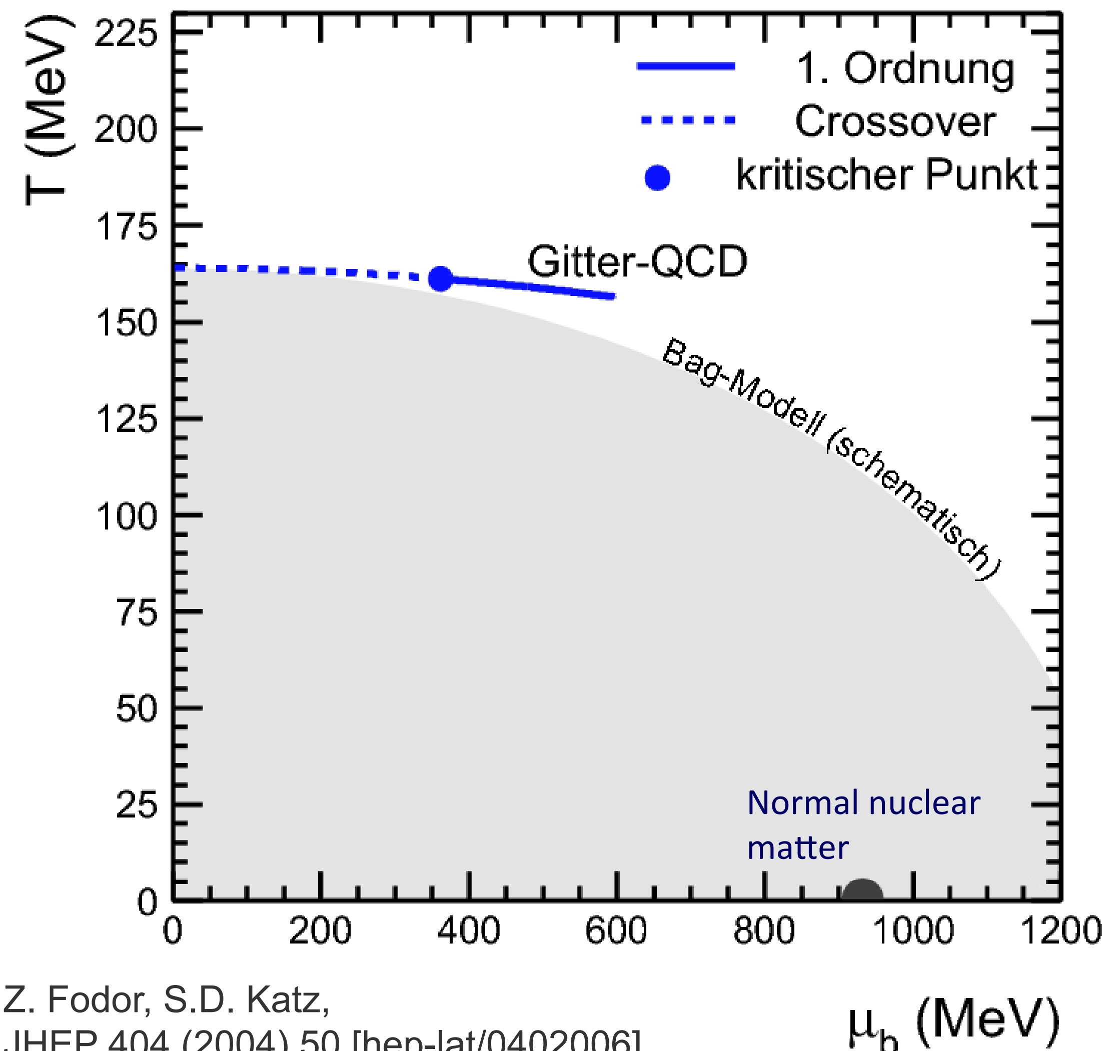
Nature of the Critical Behavior in QCD



The nature of the transition depends sensitively on the quark masses.

H. Satz,
The Thermodynamics of
Quarks and Gluons, arXiv:0803.1611

Quark-Gluon Plasma with $\mu \neq 0$: Phase Diagram from Lattice-QCD



Z. Fodor, S.D. Katz,
JHEP 404 (2004) 50 [hep-lat/0402006].

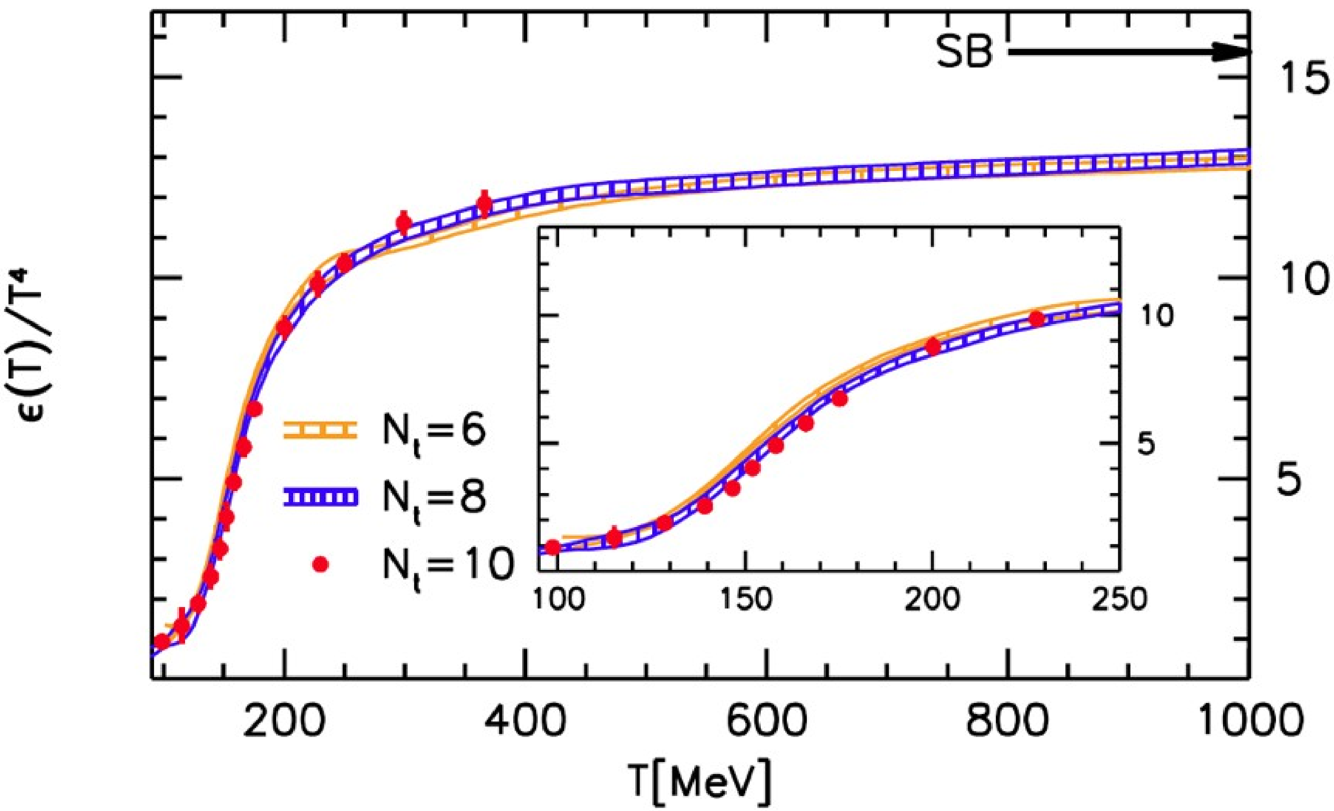
- Lattice-calculations for $\mu_b \neq 0$
  - numerically very expensive
- Some calculations suggest a critical point (with large theoretical uncertainties):
  - $T = 162$ MeV
  - $\mu_b = 340$ MeV

The existence and exact position of the critical point remains an open question

Lattice Results

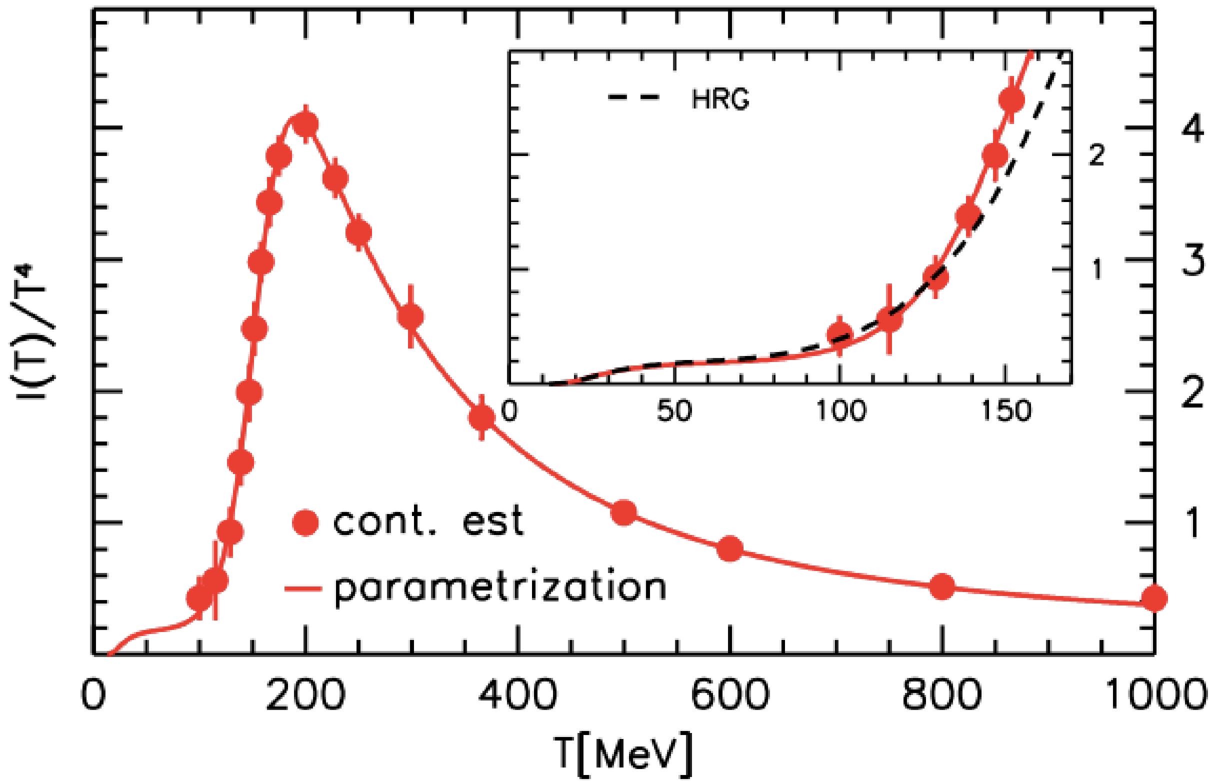
Wuppertal-Budapest coll., Borsanyi et al., JHEP 1011 (2010) 077
HotQCD collaboration, Phys.Rev. D85 (2012) 054503

- Latest lattice results for T_c (μ = 0):
  - T_c = 147 - 157 MeV (Wuppertal-Budapest collaboration)
  - T_c = 154 ± 9 MeV (HotQCD collaboration)
- Example: ε/T^4 vs. T from Wuppertal-Budapest collaboration:



Lattice QCD: "Interaction Measure"

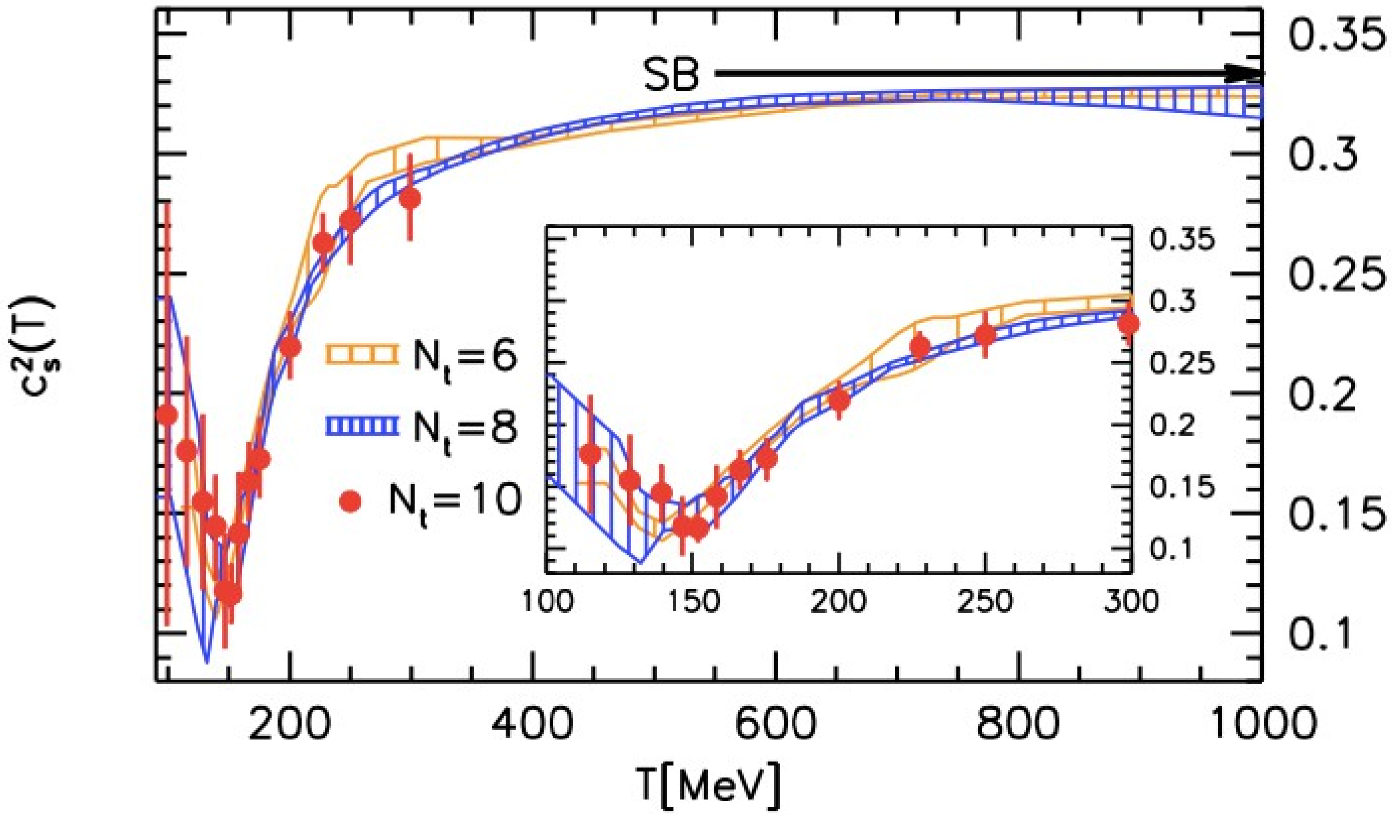
$$I(T) := \varepsilon - 3p$$



ε - 3p does not vanish around T_c as it would for an ideal gas

Lattice QCD: Sound velocity

$$c_s^2 = \frac{dp}{d\varepsilon}$$



Lattice result approaches ideal gas limit $c_s^2 = 1/3$ at a few times T_c

Points to Take Home

- When treated as a ultra-relativistic ideal gas, parameters for the transition hadron Gas ↔ QGP are:
  - T_c (μ_b=0) ≈ 150 MeV
  - $\mu_{b,c}$(T=0) = 3 $\mu_{Quark,c}$(T=0) ≈ 1.3 GeV (this is approximately five times the density of „normal“ nuclear matter)
- Lattice QCD calculations show that for temperatures up to several times T_c the assumption of an ideal gas is a poor approximation
- Transition temperature from Lattice QCD (as of 2013):
  T_c (μ_b=0) = 150 - 160 MeV