# Relativistic hydrodynamics for heavy-ion physics

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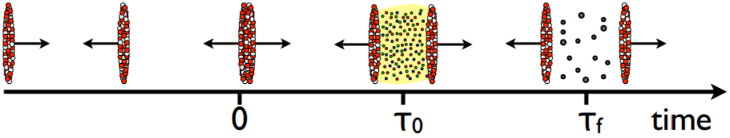
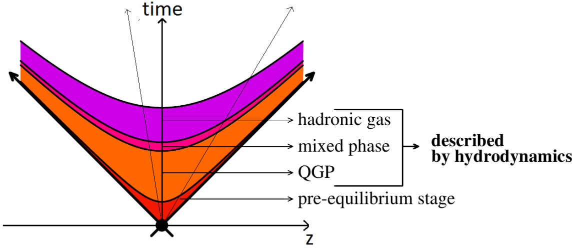
# Collision time line



# Space-time diagram

- proper time: $\tau = \sqrt{t^2 - z^2}$
- space-time rapidity $\eta_s$: $t = \tau \cosh(\eta_s)$ $z = \tau \sinh(\eta_s)$
  $\Rightarrow \eta_s = \frac{1}{2} \ln \frac{t+z}{t-z}$
- fluid rapidity $Y$: $v_z = \tanh(Y)$



# Why hydrodynamics?

- inviscid hydrodynamics describe experimental data
- $Kn \ll 1$ with $Kn \equiv \lambda / R$
  $\lambda \equiv$ mean free path; $R \equiv$ characteristic dimension of system
- one (strong) assumption: $\partial_\mu p$ and $\partial_\mu T$ small
  $\Rightarrow$ **local thermodynamic equilibrium**

# Equations of hydrodynamics

- 4-velocity $u^\mu$: $u^0 = \frac{1}{\sqrt{1-\vec{v}^2}}$ ; $u^i = \frac{v^i}{\sqrt{1-\vec{v}^2}}$
- Baryon number conservation: $\partial_\mu n^\mu = \partial_\mu (n u^\mu) = 0$
- energy-momentum tensor: $T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu - p g^{\mu\nu}$ with $T^{00}$ energy density, $T^{0j}$ momentum density, $T^{i0}$ energy flux and $T^{ij}$ momentum flux.

  In rest frame:

$$T = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

- conservation of energy and momentum: $\partial_\mu T^{\mu\nu} = 0$

# Summary (Hydrodynamic Basics)

- We have five equations: $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu(nu^\mu) = 0$
- 6 Variables: $\epsilon(x), p(x), n_B(x), u(x)$
- $\Rightarrow$ equation of state needed

Introduction

Hydrodynamics

## Thermodynamics

Hydrodynamical expansion

Summary

# Equations of thermodynamics

- differential of internal energy :
  $dU = -pdV + TdS + \mu dN$
- V, S and N extensive variables: $U = -pV + TS + \mu N$
- Gibbs-Duhem: $Vdp = SdT + Nd\mu$
- need intensive quantities $\epsilon \equiv U/V, s \equiv S/V, n \equiv N/V$:
  $\epsilon = -p + Ts + \mu n$
  $dp = sdT + nd\mu$
  $\Rightarrow d\epsilon = Tds + \mu dn$

# Velocity of sound

- velocity of sound $c_s$ is defined by: $c_s \equiv \left(\frac{\partial p}{\partial \epsilon}\right)^{\frac{1}{2}}$
- with a small disturbance $\delta\epsilon$ and $\delta p$, so that $\epsilon = \epsilon_0 + \delta\epsilon$ and $p = p_0 + \delta p$ delivers:

$$\partial_t(\delta\epsilon) + (\epsilon_0 + p_0)\nabla \cdot \vec{v} = 0$$

$$(\epsilon_0 + p_0)\frac{\partial \vec{v}}{\partial t} + \nabla \delta p = 0$$

- using definition of $c_s$ as $\delta p = {c_s}^2 \delta\epsilon$:

$$\frac{\partial^2(\delta\epsilon)}{\partial t^2} - {c_s}^2 \Delta(\delta\epsilon) = 0$$

# Equations of state for massless particles

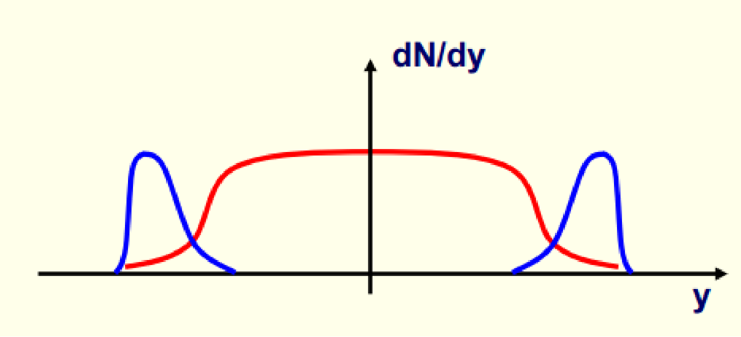
- let $\mu = 0$
- $\epsilon = g\frac{3\pi^2}{90}T^4$ [ Fermions $\epsilon = g\frac{3\pi^2}{90}T^4\frac{7}{8}$]
- $p = \frac{\epsilon}{3}$
- $n = g\frac{\pi^2}{90}T^3\frac{\zeta(3)}{\zeta(4)}$ [ Fermions $n = g\frac{\pi^2}{90}T^3\frac{7\zeta(3)}{6\zeta(4)}$ ]
- $g \approx 40$ number of degrees of freedom
  $g = 37$ [u,d] ; $g = 47.5$ [u,d,s]
- With $\epsilon + p = Ts$: $s = \frac{3nT+nT}{T} = 4n$
- Estimation of $c_s$: $c_s^2 = \frac{\partial p}{\partial \epsilon} = \frac{1}{3}$

# Bjorken Model I: Simplified model for massless particles

- Expansion only in z-direction $u^\mu = \frac{1}{\tau_0}(t, 0, 0, z)$
- $\epsilon(\tau, y) = \epsilon(\tau), p(\tau, y) = p(\tau), \beta(\tau, y) = \beta(\tau)$
  $\Rightarrow \partial_\mu T^{\mu\nu} = 0$ simplifies to $\frac{d\epsilon}{d\tau} = -\frac{\epsilon + p}{\tau} = -\frac{4}{3}\frac{\epsilon}{\tau}$
  $$\Rightarrow \boxed{\epsilon(\tau) = \epsilon(\tau_0) \cdot (\frac{\tau}{\tau_0})^{-\frac{4}{3}}}$$
- $\partial_\mu s^\mu = 0 \Rightarrow \frac{ds}{d\tau} = -\frac{s}{\tau} \Rightarrow \boxed{s(\tau) = s(\tau_0)\frac{\tau_0}{\tau}}$
- hence: $s \cdot T = \epsilon + p = \frac{4}{3}\epsilon$
  $$\Rightarrow \boxed{T(\tau) = T(\tau_0)(\frac{\tau}{\tau_0})^{-\frac{1}{3}}}$$
- Estimation of the lifetime of the QGP:
  $$\boxed{\tau_c - \tau_0 = \tau_0((\frac{T_0}{T_c})^3 - 1)}$$

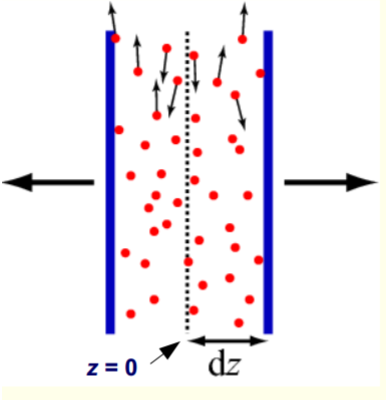
# Bjorken Model II [ more general with mass ]

- Tr $T \geq 0 \Rightarrow \epsilon \geq 3p$
  With $\frac{\partial \epsilon}{\partial \tau} = -\frac{\epsilon + p}{\tau} \Rightarrow (\frac{\tau}{\tau_0})^{-\frac{4}{3}} \geq \frac{\epsilon(\tau)}{\epsilon(\tau_0)} \geq (\frac{\tau}{\tau_0})^{-1}$
- $n(\tau, y) = n(\tau)$ leads to constant rapidity density
  $\frac{dN}{dy} = const$
- initial condition $\epsilon(\tau_0)\tau_0 = ?$



# Estimation of the energy density

- $dE = dN < m_T \cdot \cosh(y) >|_{y=0}$ ; $[\ \frac{dy}{dz}|_{z=0} = \frac{d}{dz}\text{atanh}(\frac{z}{\tau})|_{z=0} = \frac{1}{\tau}]$
- $\epsilon_0 = \frac{<m_T>}{A}\frac{dN}{dz} = \frac{<m_T>}{A}\frac{dN}{dy}\frac{dy}{dz}|_{y=0} = \frac{<m_T>}{A\tau_0}\frac{dN}{dy}$

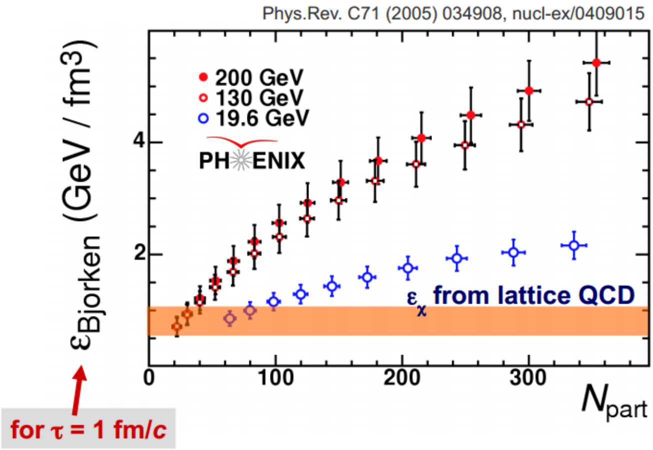



- $$\boxed{\epsilon_0\tau_0 = \frac{1}{A}\frac{dE_T}{dy}}$$

# Estimation of the energy density

- $A \approx \pi R_{Pb}^2 \approx 140\text{fm}^2$
- $\frac{dN}{dy}\mid_{y=0} \approx 2450 \pm 92$ from (Pb+Pb@$\sqrt{s_{NN}} = 2.76\text{TeV}$)
- $\epsilon_0 \tau_0 \approx (11.7 \pm 0.43)\frac{\text{GeV}}{\text{fm}^2}$
- for an initial time of $\tau_0 \approx 1\text{fm}$ this leads to $\epsilon_0 \approx 10\frac{\text{GeV}}{\text{fm}^3}$
- from $\epsilon_0 = g\frac{\pi^2}{30}T_0^4$ we get initial temperature $T_0 \approx 280\text{MeV}$ and with $T_c \approx 170\text{MeV}$ the livetime $\tau_c - \tau_0 \approx 3.6\frac{\text{fm}}{\text{c}}$

# Bjorken energy density from data



# Longitudinal acceleration

- more general distribution of $s(\tau, x, y, \eta_s)$
- acceleration $x$,$y$,$z$-direction
- Assumption:

$$s(x, y, \eta_s) \sim \exp(-\frac{1}{2}((\frac{x}{\sigma_x})^2 + (\frac{y}{\sigma_y})^2 + (\frac{\eta}{\sigma_\eta})^2)$$

- from Euler-equation we get:

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\epsilon + p}\frac{\partial p}{\partial z} = -c_s^2 \frac{\partial \ln(s)}{\partial z}$$

- $Y(\tau) = (1 + \frac{c_s^2 \ln \frac{\tau}{\tau_0}}{\sigma_\eta^2})\eta_s$
- $\sigma_\eta \to \infty$ leads to the Bjorkencondition

# Transversal acceleration

- $\frac{\partial v_x}{\partial t} = -\frac{1}{\epsilon + p}\frac{\partial p}{\partial x} = -c_s^2 \frac{\partial \ln(s)}{\partial x}$
- $v_x = c_s^2 \frac{x}{\sigma_x^2} t$
- $v_y = c_s^2 \frac{y}{\sigma_y^2} t$
- $\sigma_x < \sigma_y$ non-central collision
- $\frac{dN}{d\Phi} = 1 + 2v_2 cos(2\Phi),\ v_2 \equiv$ elliptic flow
- $v_2 \sim \epsilon \equiv \frac{\sigma_y^2 - \sigma_x^2}{\sigma_y^2 + \sigma_x^2}$

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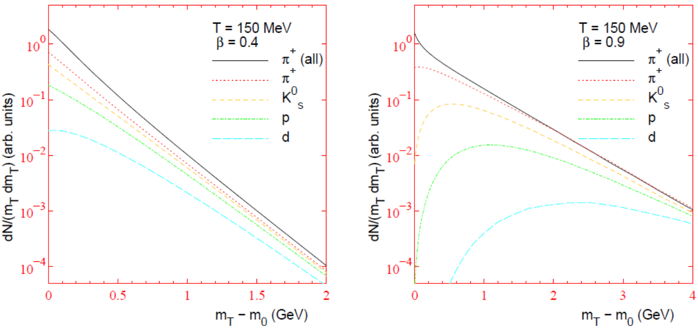
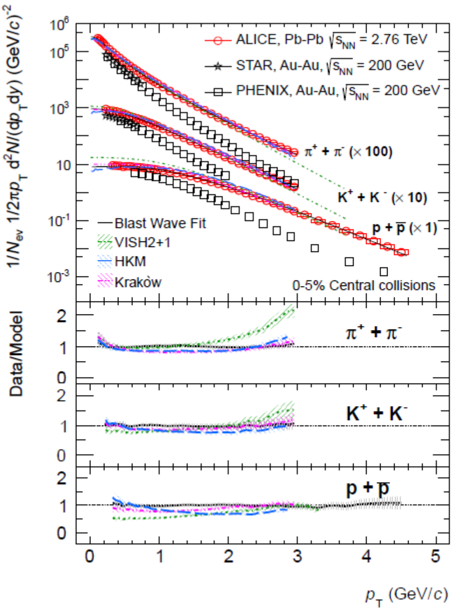



- radial flow

  $u = \sqrt{u_1 u^1 + u_2 u^2}$ ; $m_0^2 = m_t^2 - p_t^2$

  $\frac{d}{dm_t} \log \frac{dN}{2\pi p_t dp_t dp_z} \propto \frac{-u_0 + u m_t / p_t}{T}$

# Summary



ALICE, Pb-Pb $\sqrt{s_{NN}}$ = 2.76 TeV
STAR, Au-Au, $\sqrt{s_{NN}}$ = 200 GeV
PHENIX, Au-Au, $\sqrt{s_{NN}}$ = 200 GeV
$\pi^+ + \pi^-$ (× 100)
$K^+ + K^-$ (× 10)
$p + \bar{p}$ (× 1)
Blast Wave Fit
VISH2+1
HKM
Kraków
0-5% Central collisions
$1/N_{ev}\ 1/2\pi p_T\ d^2N/(dp_T dy)\ (\mathrm{GeV}/c)^{-2}$
Data/Model
$p_T$ (GeV/c)

# Summary

- hydrodynamics describe experimental data
- simplifications enable one to solve hydrodynamic equations
- we can estimate the order of magnitude of $\epsilon$, $p$ and $T$ as a function of time

# References


- J-Y Ollitraut, Relativistic hydrodynamics for heavy-ion collision, Eur.J.Phys.29:275-302,2008, arXiv:0708.2433v2
- J.D. Bjorken, Highly relativistic nucleus-nucleus collision: The central rapidity region, Phys. Rev. D27 1 (1983)