

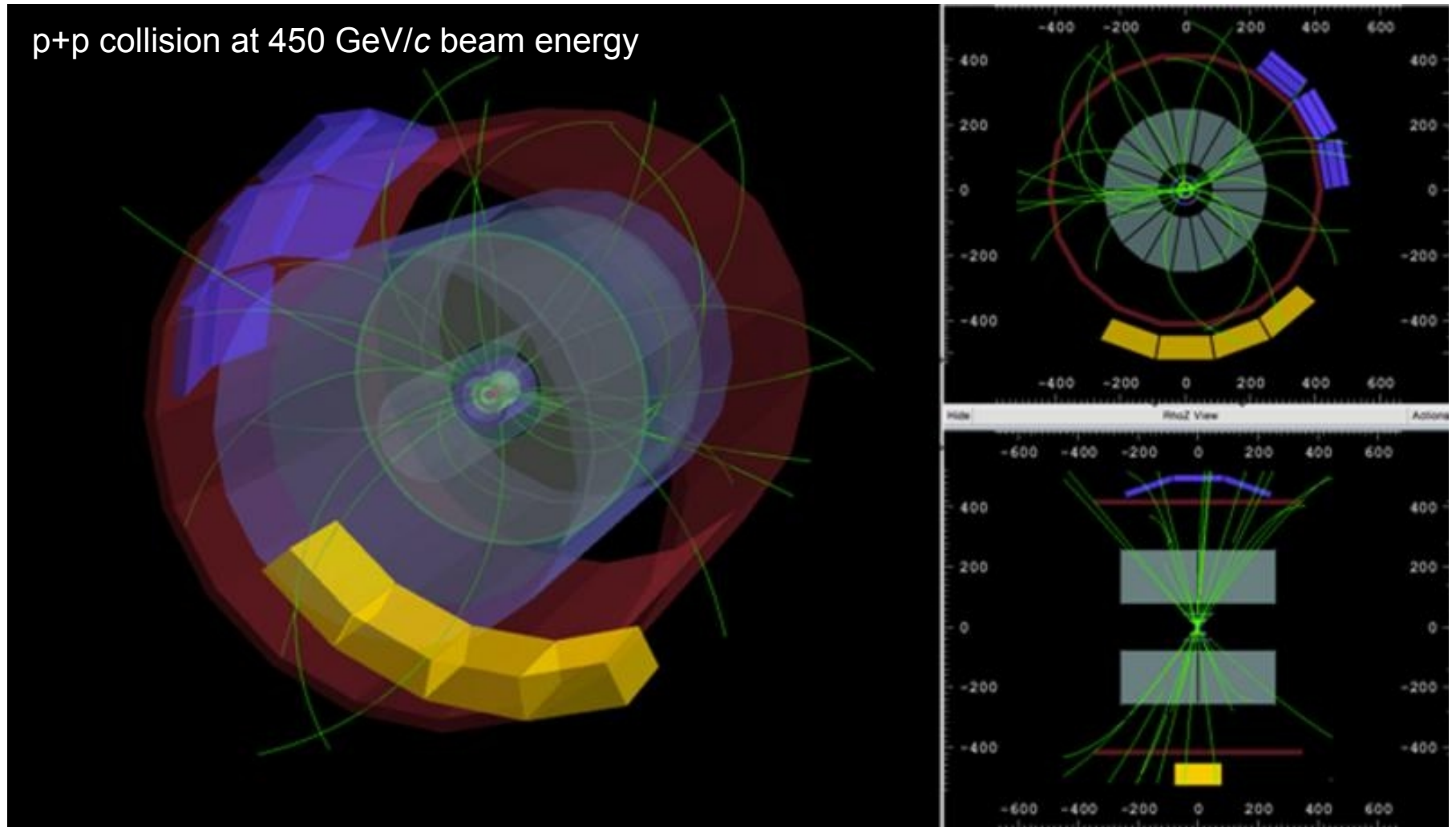
QGP Physics – from Fixed Target to LHC

2. Kinematic Variables

Prof. Dr. Klaus Reygers, Prof. Dr. Johanna Stachel
Physikalisches Institut, Universität Heidelberg
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First collisions at the LHC after two years of maintenance



From June 2015 on: p+p collisions at 6500 GeV/c beam energy

Notations and Conventions

Natural units: $c = \hbar = 1$ also: $k_B = 1 \rightarrow E = k_B T, T = 2 \cdot 10^{12} \text{K} \approx 172 \text{MeV}$

useful: $1 = \hbar c = 0.197 \text{GeV} \cdot \text{fm}$

Space-time coordinates
(contravariant vector): $x^\nu = (x^0, x^1, x^2, x^3) = (t, \vec{x}) = (t, x, y, z)$

Relativistic energy and momentum:

$$E = \gamma m, \quad p = \gamma \beta m, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad m = \text{rest mass}$$


4-momentum vector: $p^\mu = (p^0, p^1, p^2, p^3) = (E, \vec{p}) = (E, \vec{p}_T, p_z) = (E, p_x, p_y, p_z)$

Scalar product of two 4-vectors a and b : $a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$

Relation between energy and momentum: $E^2 = p^2 + m^2$

Center-of-Momentum System (CMS)

Consider a collision of two particles. The CMS is defined by $\vec{p}_a = -\vec{p}_b$

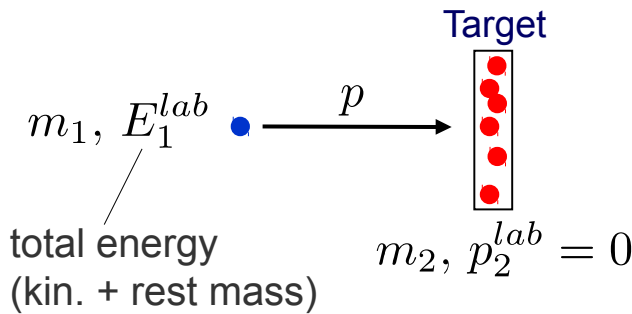
$$p_a = (E_a, \vec{p}_a) \qquad p_b = (E_b, \vec{p}_b)$$


The Mandelstam variable s is defined as $s := (p_a + p_b)^2 \stackrel{CMS}{=} (E_a + E_b)^2$

The center-of-mass energy \sqrt{s} is the total energy available in the CMS

√s for Fixed-Target und Collider Experiments (I)

Fixed-target experiment:



$$\begin{aligned}
 s &= \left[\begin{pmatrix} E_1^{lab} \\ \vec{p}_1 \end{pmatrix} + \begin{pmatrix} m_2 \\ \vec{0} \end{pmatrix} \right]^2 \\
 &= E_1^{lab^2} + 2E_1^{lab}m_2 + m_2^2 - p_1^2 \\
 &= m_1^2 + m_2^2 + 2E_1^{lab}m_2
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \sqrt{s} &= \sqrt{m_1^2 + m_2^2 + 2E_1^{lab}m_2} \\
 &\stackrel{E_1^{lab} \gg m_1, m_2}{\approx} \sqrt{2E_1^{lab}m_2}
 \end{aligned}$$

Example: antiproton production

(fixed-target experiment): $p + p \rightarrow p + p + p + \bar{p}$

Minimum energy required to produce an antiproton:

In CMS, all particles at rest after the reaction, i.e., $\sqrt{s} = 4 m_p$, hence:

$$4m_p \stackrel{!}{=} \sqrt{2m_p^2 + 2E_1^{lab, \min} m_p} \Rightarrow E_1^{lab, \min} = \frac{(4m_p)^2 - 2m_p^2}{2m_p} = 7m_p$$

\sqrt{s} for Fixed-Target und Collider Experiments (II)

Collider:



$$s = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

$$= m_1^2 + m_2^2 + 2E_1 E_2 - 2\vec{p}_1 \vec{p}_2$$

for $\vec{p}_1 = -\vec{p}_2$ and $m_1 = m_2$: $\sqrt{s} = 2E$ where $E \equiv E_1 = E_2$

Example: LHC

1232 dipoles x 14.3 m / (2 π)

rigidity $R = \frac{p}{q} = r_{\text{LHC,bend}} \cdot B_{\text{LHC}}$, $B_{\text{LHC,max}} \approx 8.3 \text{ T}$ limits \sqrt{s}

protons $R = p_{\text{proton}}$ ions: $R = \frac{A \cdot p_{\text{nucleon}}}{Z}$

2011/12: $p_{\text{proton}} = 3.5 \text{ TeV} \rightarrow p_{\text{nucleon}} \equiv p_{\text{Pb}}/A = \frac{Z}{A} \cdot p_{\text{proton}} = 1.38 \text{ TeV}$

Center-of-momentum energy per nucleon-nucleon pair:

2011/12 Pb beam: $\sqrt{s_{NN}} = 2.76 \text{ TeV}$

2015 Pb beam: $\sqrt{s_{NN}} = 5.1 \text{ TeV}$

Rapidity

The rapidity y is a generalization of the (longitudinal) velocity $\beta_L = p_L/E$:

$$y := \operatorname{arctanh} \beta_L = \frac{1}{2} \ln \frac{1 + \beta_L}{1 - \beta_L} = \frac{1}{2} \ln \frac{E + p_L}{E - p_L}$$

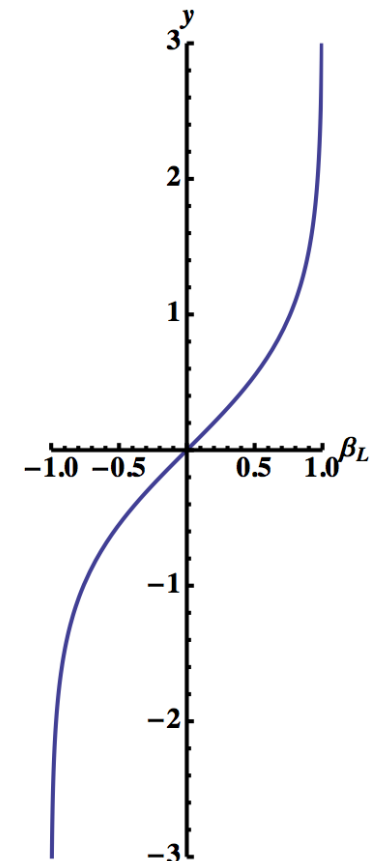
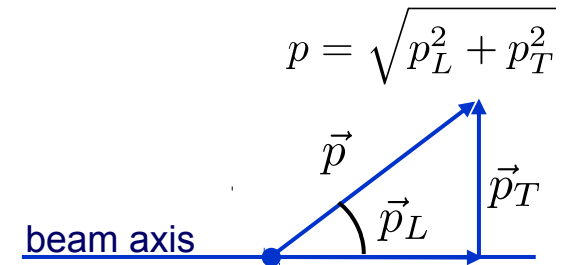
$$y \approx \beta_L \text{ for } \beta_L \ll 1$$

With $e^y = \sqrt{\frac{E + p_L}{E - p_L}}$, $e^{-y} = \sqrt{\frac{E - p_L}{E + p_L}}$

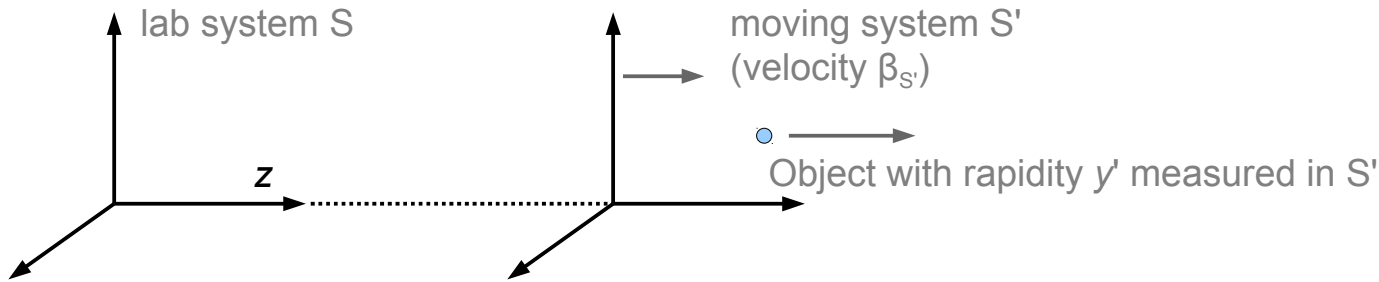
and $\sinh x = \frac{1}{2} (e^x - e^{-x})$, $\cosh x = \frac{1}{2} (e^x + e^{-x})$

we readily obtain $E = m_T \cdot \cosh y$, $p_L = m_T \cdot \sinh y$

where $m_T := \sqrt{m^2 + p_T^2}$ is called the *transverse mass*



Additivity of Rapidity under Lorentz Transformation



Lorentz transformation: $E = \gamma(E' + \beta p'_z), \quad p_z = \gamma(p'_z + \beta E') \quad (\beta \equiv \beta_{S'})$

$$\begin{aligned}
 y &= \frac{1}{2} \ln \frac{E + p_z}{E - p_z} \\
 &= \frac{1}{2} \ln \frac{\gamma(E' + \beta p'_z) + \gamma(p'_z + \beta E')}{\gamma(E' + \beta p'_z) - \gamma(p'_z + \beta E')} \\
 &= \frac{1}{2} \ln \frac{(1 + \beta)(E' + p'_z)}{(1 - \beta)(E' - p'_z)} \\
 &= \underbrace{\frac{1}{2} \ln \frac{1 + \beta}{1 - \beta}}_{\text{rapidity of S' as measured in S}} + \underbrace{\frac{1}{2} \ln \frac{E' + p'_z}{E' - p'_z}}_{y'}
 \end{aligned}$$

y is not Lorentz invariant, however, it has a simple transformation property:

$$y = y' + y_{S'}$$

Rapidity of the CMS

Consider collisions of two particles with equal mass m and rapidities y_a and y_b . The rapidity of the CMS y_{CM} is then given by:


$$y_{CM} = (y_a + y_b)/2$$

In the center-of-mass frame, the rapidities of a and b are:

$$y_a^* = -(y_b - y_a)/2 \quad \text{and} \quad y_b^* = (y_b - y_a)/2$$

Examples (CMS rapidity of the nucleon-nucleon system)

a) fixed target experiment: $y_{CM} = (y_{\text{target}} + y_{\text{beam}})/2 = y_{\text{beam}}/2$

b) collider (same species and beam momentum): $y_{CM} = (y_{\text{target}} + y_{\text{beam}})/2 = 0$

c) collider (two different ions species, same rigidity):

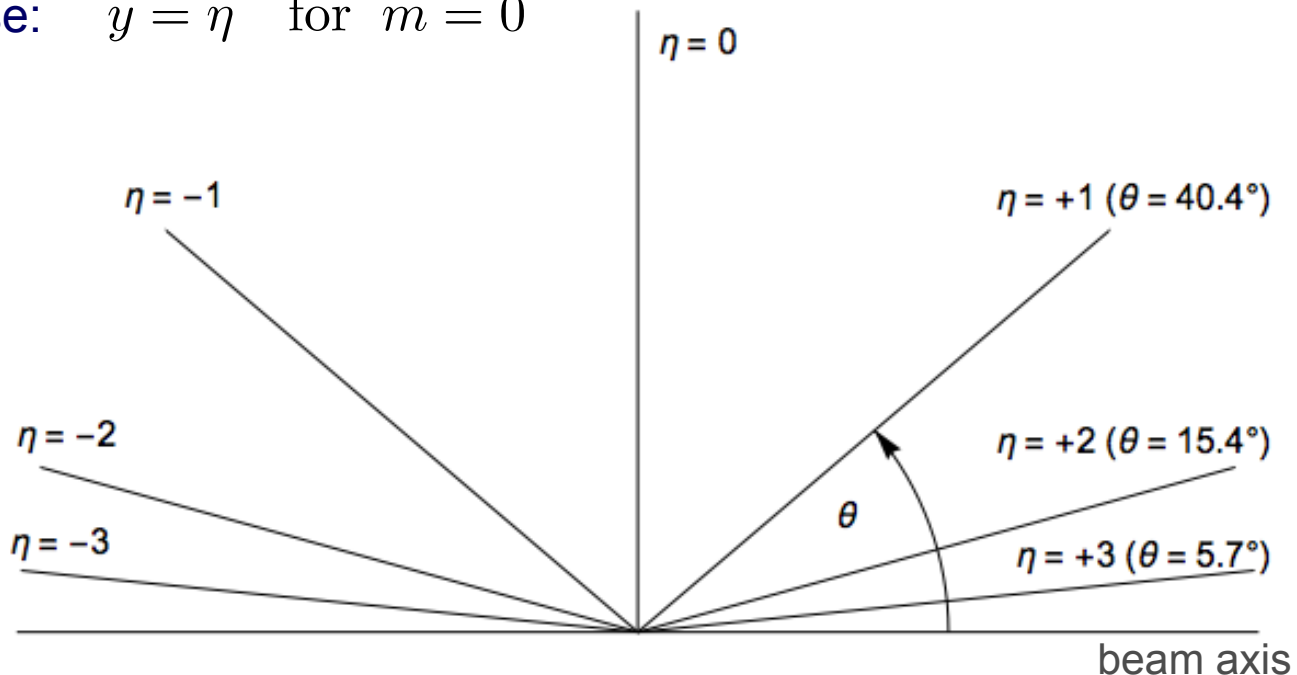
$$y_{CM} = \frac{1}{2} \ln \frac{Z_1 A_2}{A_1 Z_2} \quad \text{[exercise]} \quad \text{example: p-Pb: } y_{CM} \approx 0.465$$

Pseudorapidity

$$y = \frac{1}{2} \ln \frac{E + p \cos \vartheta}{E - p \cos \vartheta} \stackrel{p \gg m}{\approx} \frac{1}{2} \ln \frac{1 + \cos \vartheta}{1 - \cos \vartheta} = \frac{1}{2} \ln \frac{2 \cos^2 \frac{\vartheta}{2}}{2 \sin^2 \frac{\vartheta}{2}} = -\ln \left[\tan \frac{\vartheta}{2} \right] =: \eta$$

$$\cos(2\alpha) = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

Special case: $y = \eta$ for $m = 0$



Analogous to the relations for the rapidity we find:

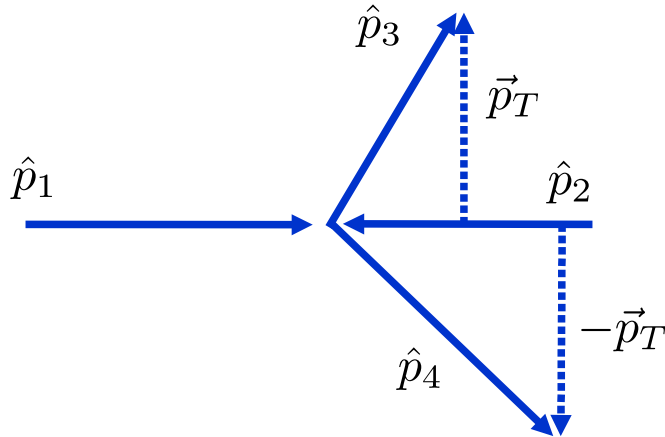
$$p = p_T \cdot \cosh \eta, \quad p_L = p_T \cdot \sinh \eta$$

Example: Beam Rapidities

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \ln \frac{E + p_z}{\sqrt{E^2 - p_z^2}} = \ln \frac{E + p_z}{m} \approx \ln \frac{2E}{m}$$

Beam momentum (GeV/c)	Beam rapidity
100	5.36
158	5.81
1380 (= 3500·82/208)	7.99
2562 (= 6500·82/208)	8.61
2760 (= 7000·82/208)	8.86
3500	8.92
6500	9.54
7000	9.61

2-to-2 Hard Scattering of Massless Partons



beam energy

$$P_1 = (E, \vec{p}), \quad E \approx |\vec{p}| \approx \sqrt{s}/2, \quad P_2 = (E, -\vec{p})$$

$$\hat{p}_1 = x_1 P_1, \quad \hat{p}_2 = x_2 P_2$$

Bjorken x of the partons

$$\hat{p}_3 = (p_T \cosh y_3, \vec{p}_T, p_T \sinh y_3)$$

$$\hat{p}_4 = (p_T \cosh y_4, -\vec{p}_T, p_T \sinh y_4)$$

Calculate Bjorken x of the two partons (1 and 2):

$$\hat{p}_1 + \hat{p}_2 = \hat{p}_3 + \hat{p}_4$$



$$(x_1 + x_2) \frac{\sqrt{s}}{2} = p_T (\cosh y_3 + \cosh y_4)$$

$$(x_1 - x_2) \frac{\sqrt{s}}{2} = p_T (\sinh y_3 + \sinh y_4)$$

$$x_1 = \frac{p_T}{\sqrt{s}} (e^{y_3} + e^{y_4})$$

$$x_2 = \frac{p_T}{\sqrt{s}} (e^{-y_3} + e^{-y_4})$$

$$y_3 = y_4 = 0: \quad x_1 = x_2 = \frac{2p_T}{\sqrt{s}} =: x_T$$

Quick Overview: Kinematic Variables

Transverse momentum

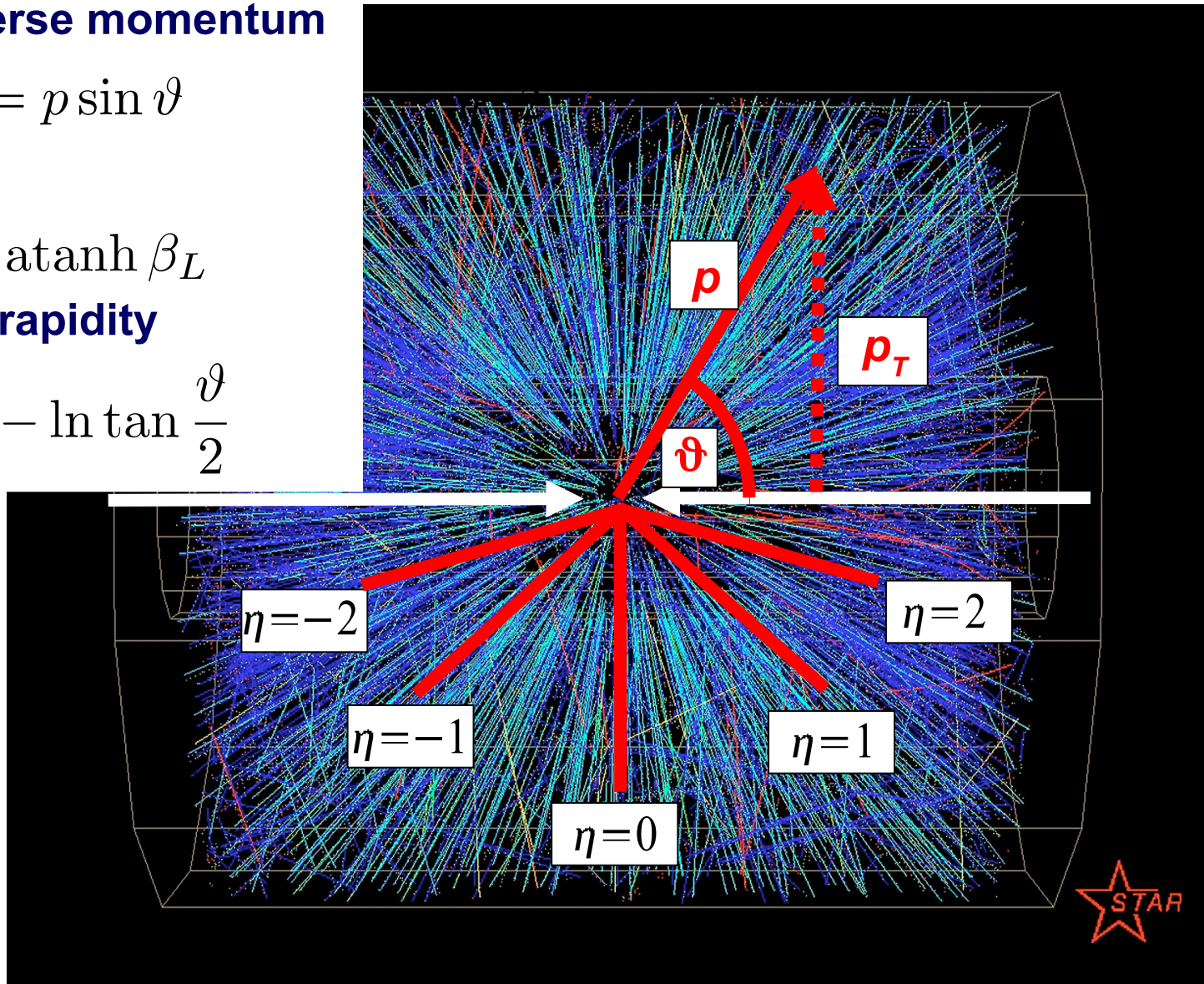
$$p_T = p \sin \vartheta$$

rapidity

$$y = \operatorname{atanh} \beta_L$$

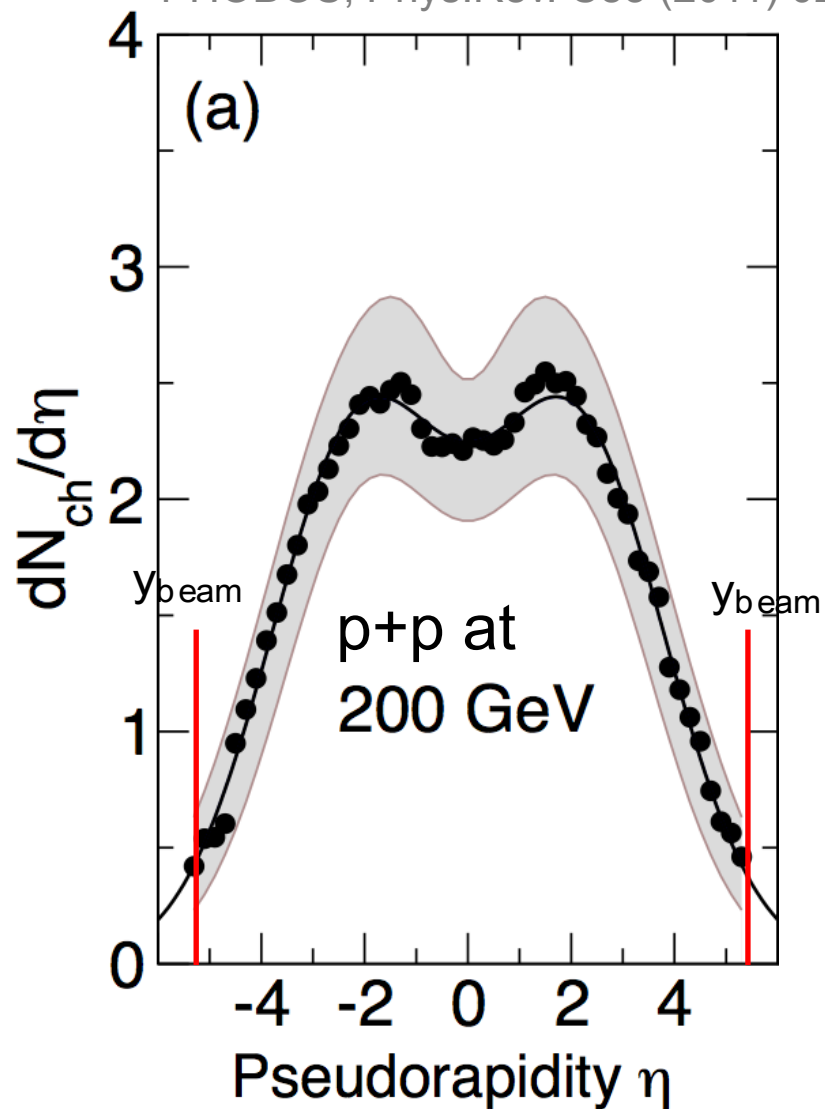
pseudorapidity

$$\eta = -\ln \tan \frac{\vartheta}{2}$$



Example of a Pseudorapidity Distribution

PHOBOS, Phys.Rev. C83 (2011) 024913



Beam rapidity:

$$y_{beam} = \ln \frac{E+p}{m} = 5.36$$

Average number of charged particles per collision:

$$\langle N_{ch} \rangle = \int \frac{dN_{ch}}{d\eta} d\eta \approx 20$$

Difference between dN/dy and $dN/d\eta$ in the CMS

$$\frac{dN}{d\eta} = \sqrt{1 - \frac{m^2}{m_T^2 \cosh^2 y}} \frac{dN}{dy}$$

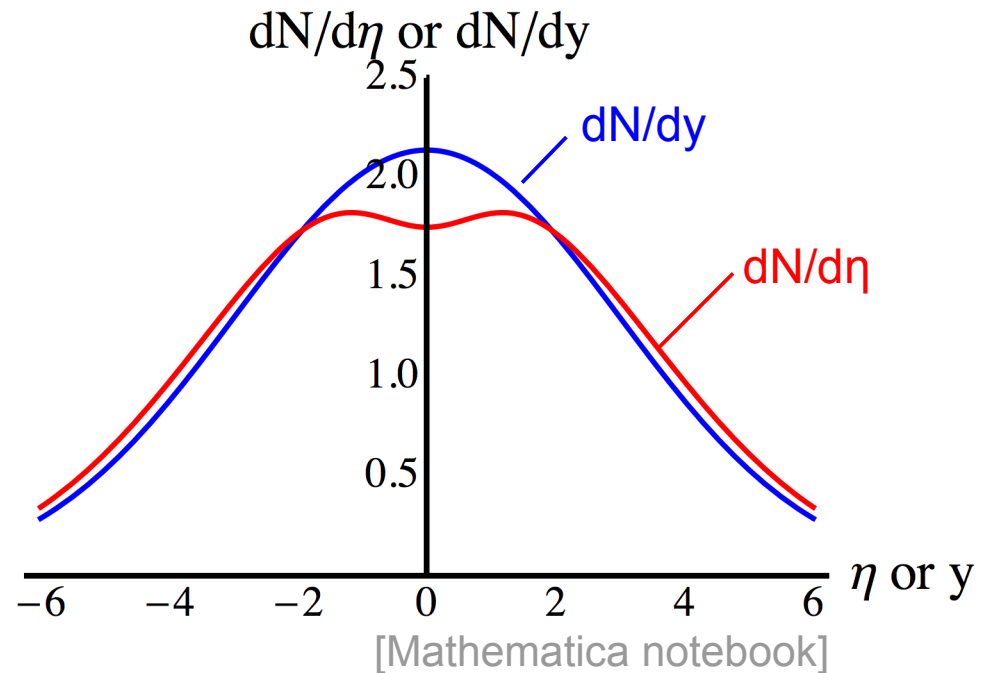
$$y(\eta) = \frac{1}{2} \log \left(\frac{\sqrt{p_T^2 \cosh^2(\eta) + m^2} + p_T \sinh(\eta)}{\sqrt{p_T^2 \cosh^2(\eta) + m^2} - p_T \sinh(\eta)} \right)$$

Difference between dN/dy and $dN/d\eta$
in the CMS at $y = 0$:

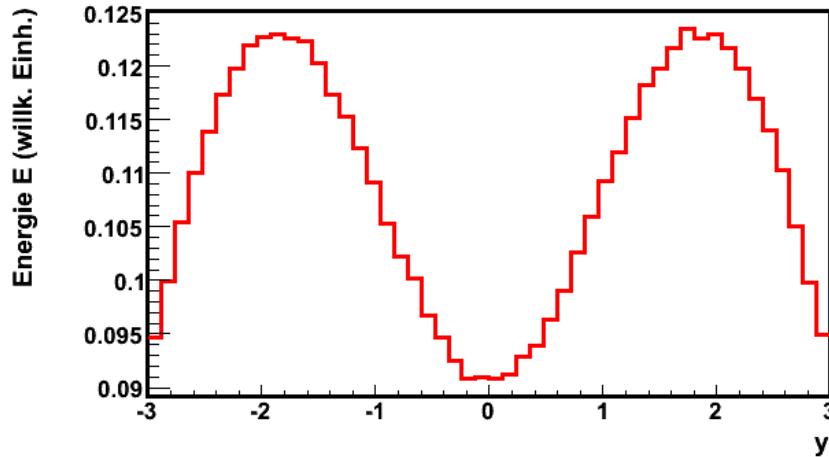
Simple example:
Pions distributed according to

$$\frac{1}{2\pi p_T} \frac{d^2 N}{dp_T dy} = G(y) \cdot \exp(-p_T/0.16)$$

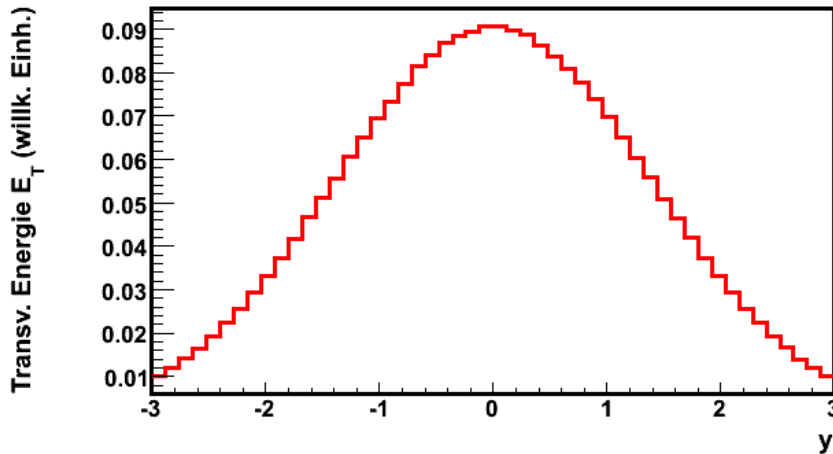
Gaussian with $\sigma=3$



Total Energy and Transverse Energy



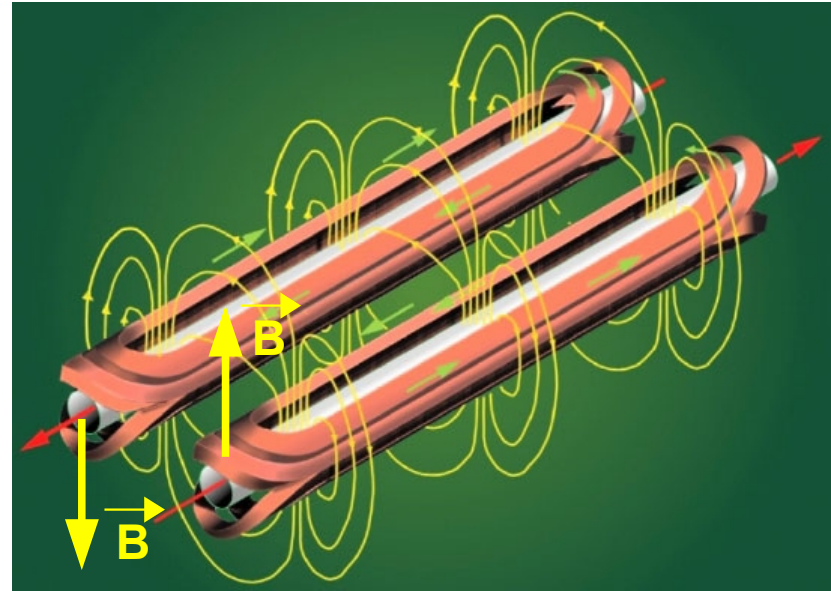
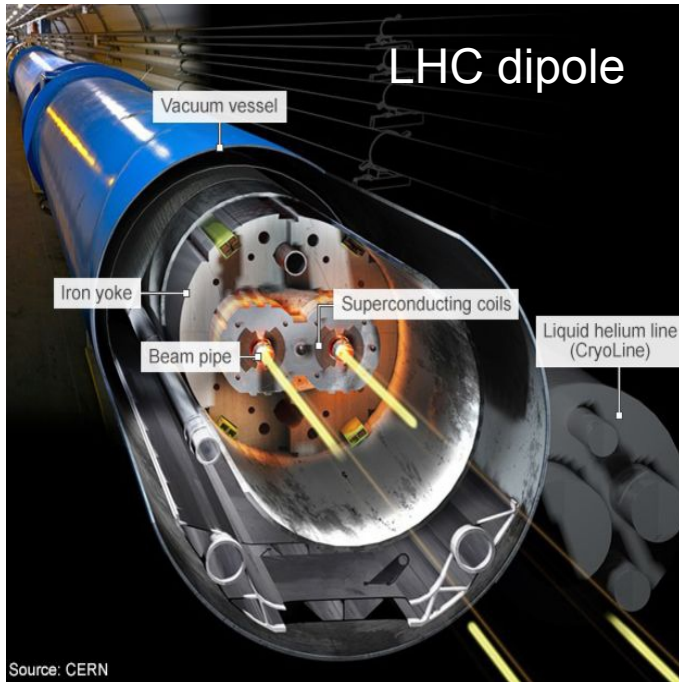
$$\leftarrow \frac{dE}{dy} \quad E = \sqrt{m^2 + p^2}$$



$$\leftarrow \frac{dm_T}{dy} \quad m_T = \sqrt{m^2 + p_T^2}$$

Total energy concentrated at forward (i.e., small) angles

Luminosity and Cross Sections (I)



transverse beam radius: about 20 μm

	pp 2011	Pb-Pb 2011
Beam energy (per nucleon)	3.5 TeV	3.5 TeV · 82/208
Particles/bunch	$1.35 \cdot 10^{11}$	$1.2 \cdot 10^8$
#bunches per beam	1380	358
Bunch spacing	50 ns (= 15 m)	200 ns
RMS bunch length	7.6 cm	9.8 cm
peak luminosity	$3.65 \cdot 10^{33} \text{ cm}^{-2} \text{ s}^{-1}$	$0.5 \cdot 10^{27} \text{ cm}^{-2} \text{ s}^{-1}$

Luminosity and Cross Sections (II)

The luminosity L of a collider is defined by:

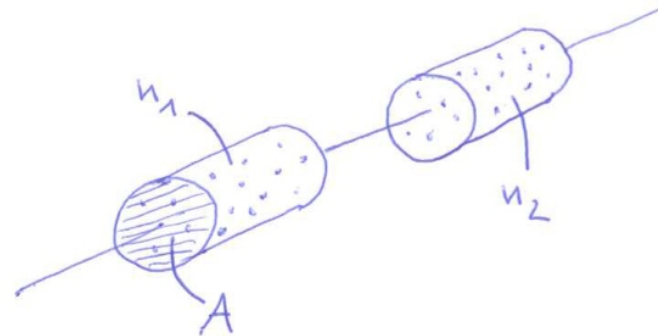
$$\frac{dN_{\text{int}}}{dt} = \sigma \cdot L$$

L = luminosity (in $\text{s}^{-1}\text{cm}^{-2}$)

dN_{int}/dt = Number of interactions of a certain type per second

σ = cross section for this reaction

$$L = \frac{n_1 n_2 f_{\text{coll}}}{A}$$



n_1, n_2 = numbers of particles per bunch in the two beams

f_{coll} = bunch collision frequency at a given crossing point

A = beam crossing area

Luminosity and Cross Sections (III)

The luminosity can be determined by measuring the beam current:

$$I_{1,2} = n_{1,2} \cdot N_b \cdot e \cdot f_{\text{rev}}$$

N_b = number of bunches in the beam

e = elementary electric charge

f_{rev} = revolution frequency

The crossing area A is usually calculated as

$$A = 4\pi\sigma_x\sigma_y$$

The standard deviations of the beam profiles are measured by sweeping the beams transversely across each other in a so called van der Meer scan.

Integrated luminosity:
$$L_{\text{int}} = \int L dt$$

Lorentz invariant Phase Space Element

If one is interested in the production of a particle A one could define the observable

$$\frac{1}{L_{\text{int}}} \frac{d^3 N_A}{d^3 \vec{p}} = \frac{1}{L_{\text{int}}} \frac{d^3 N_A}{dp_x dp_y dp_z}$$

However, the phase space density would then not be Lorentz invariant:

$$\frac{d^3 N}{dp'_x dp'_y dp'_z} = \frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} \cdot \frac{d^3 N}{dp_x dp_y dp_z} = \frac{E}{E'} \cdot \frac{d^3 N}{dp_x dp_y dp_z}$$

We thus use the Lorentz invariant phase space element $\frac{d^3 \vec{p}}{E} = \frac{dp_x dp_y dp_z}{E}$

The corresponding observable is called Lorentz invariant cross section:

$$E \frac{d^3 \sigma}{d^3 \vec{p}} = \frac{1}{L_{\text{int}}} E \frac{d^3 N}{d^3 \vec{p}} = \underbrace{\frac{1}{N_{\text{evt,tot}}} E \frac{d^3 N}{d^3 \vec{p}}}_{\text{this is called the invariant yield}} \sigma_{\text{tot}}$$

[Lorentz invariant Phase Space Element: Proof of Invariance]

Lorentz boost along the z axis:

$$p'_x = p_x$$

$$p'_y = p_y$$

$$p'_z = \gamma(p_z - \beta E), \quad p_z = \gamma(p'_z + \beta E')$$

$$E' = \gamma(E - \beta p_z), \quad E = \gamma(E' + \beta p'_z)$$

Jacobian:
$$\frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} = \begin{vmatrix} \frac{\partial p_x}{\partial p'_x} & 0 & 0 \\ 0 & \frac{\partial p_y}{\partial p'_y} & 0 \\ 0 & 0 & \frac{\partial p_z}{\partial p'_z} \end{vmatrix}$$

$$\frac{\partial p_x}{\partial p'_x} = 1, \quad \frac{\partial p_y}{\partial p'_y} = 1, \quad \frac{\partial p_z}{\partial p'_z} = \frac{\partial}{\partial p'_z} [\gamma(p'_z + \beta E')] = \gamma \left(1 + \beta \frac{\partial E'}{\partial p'_z} \right)$$

$$\frac{\partial E'}{\partial p'_z} = \frac{\partial}{\partial p'_z} \left[(m^2 + p_x'^2 + p_y'^2 + p_z'^2)^{1/2} \right] = \frac{p'_z}{E'}$$

$$\rightsquigarrow \frac{\partial p_z}{\partial p'_z} = \gamma \left(1 + \beta \frac{p'_z}{E'} \right) = \frac{E}{E'}$$

And so we finally obtain:
$$\frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} = \frac{E}{E'}$$

Invariant Cross Section

Invariant cross section in practice:

$$E \frac{d^3\sigma}{d^3p} = E \frac{1}{p_T} \frac{d^3\sigma}{dp_T dp_z d\varphi}$$

$$\stackrel{dp_z/dy = m_T \cosh y = E}{=} \frac{1}{p_T} \frac{d^3\sigma}{dp_T dy d\varphi}$$

$$\stackrel{\text{symmetry in } \varphi}{=} \frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T dy}$$

Sometimes also measured as a function of m_T :

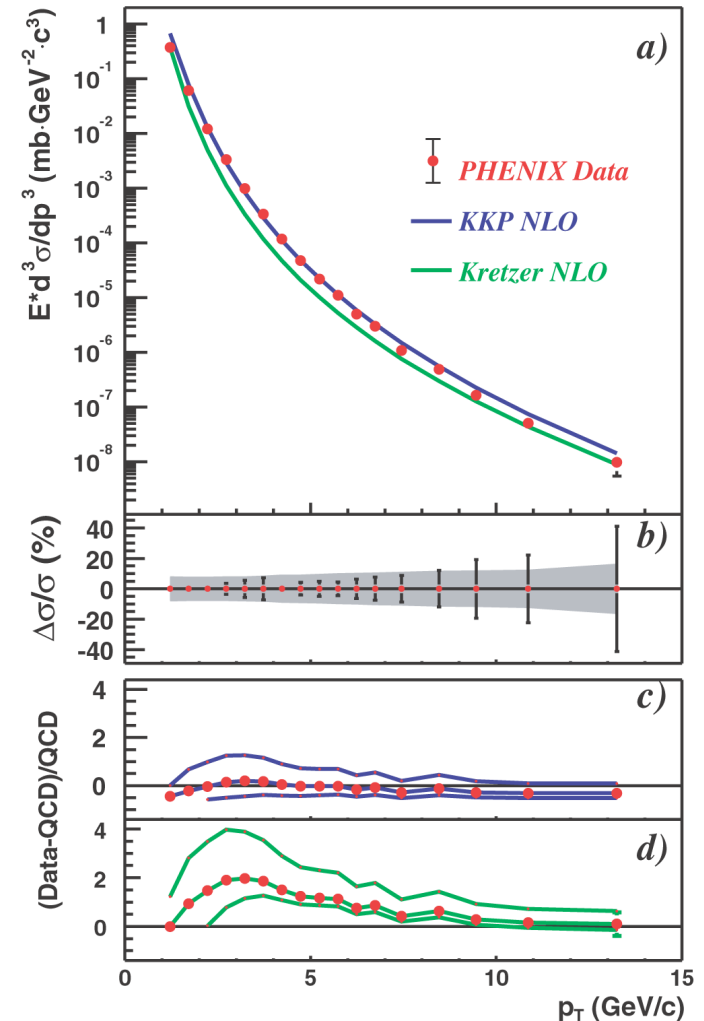
$$\frac{1}{2\pi m_T} \frac{d^2\sigma}{dm_T dy} = \frac{1}{2\pi m_T} \frac{d^2\sigma}{dp_T dy} \frac{dp_T}{dm_T} = \frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T dy}$$

Integral of the inv. cross section:

$$\int E \frac{d^3\sigma}{d^3p} d^3p / E = \langle N_x \rangle \cdot \sigma_{\text{tot}}$$

Average yield of particle X per event

Example: Invariant cross section for neutral pion production in p+p at $\sqrt{s} = 200$ GeV



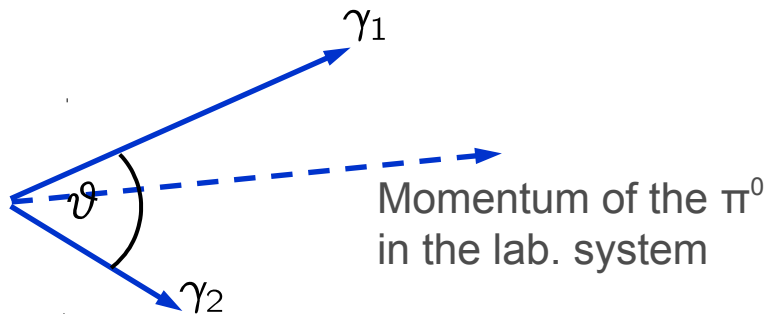
Invariant Mass

Consider the decay of a particle in two daughter particles. The mass of the mother particle is given by (“invariant mass”):

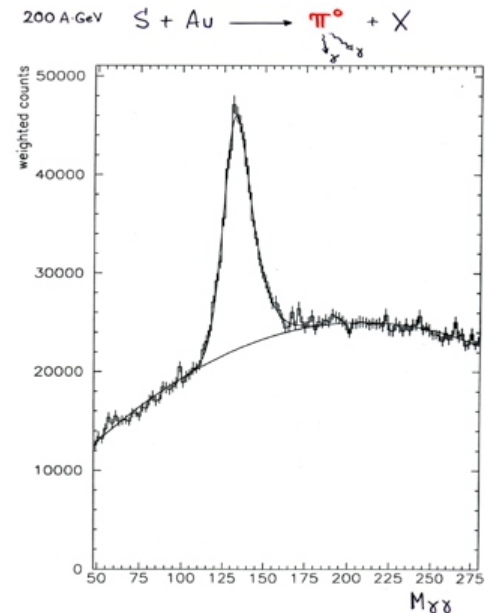
$$\begin{aligned}
 M^2 &= \left[\begin{pmatrix} E_1 \\ \vec{p}_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ \vec{p}_2 \end{pmatrix} \right]^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \\
 &= m_1^2 + m_2^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2 \\
 &= m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2 \cos \vartheta
 \end{aligned}$$

Example: π^0 decay: $\pi^0 \rightarrow \gamma + \gamma$, $m_1 = m_2 = 0$, $E_i = p_i$

$$\Rightarrow M = \sqrt{2E_1E_2(1 - \cos \vartheta)}$$



Example:
 π^0 peak with
combinatorial
background



Points to Take Home

Center-of-mass energy \sqrt{s} : Total energy in the center-of-mass (or momentum) system (rest mass of + kinetic energy)

Observables: Transverse momentum p_T and rapidity y

Pseudorapidity $\eta \approx y$ for $E \gg m$ ($\eta = y$ for $m = 0$, e.g., for photons)

Production rates of particles described by the Lorentz invariant cross section:

Lorentz-invariant cross section: $E \frac{d^3\sigma}{d^3p}$