

QGP Physics – from Fixed Target to LHC

3. Thermodynamics of the QGP

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3.1 QGP in the MIT bag model

Thermodynamics of

relativistic Bose gas

relativistic Fermi-gas

Bag model of hadrons

Constructing the phase diagram between pion gas and QGP

“

realistic hadron gas and QGP

3.1.1 Thermodynamics of a relativistic Bose gas

probability density for occupation of state with relativistic energy E and degeneracy g

$$N(E) = \frac{g}{(2\pi)^3} \left(\exp\left(\frac{E - \mu}{T}\right) - 1 \right)^{-1}$$

with energy $E^2 = p^2 + m^2$

(note: here $\hbar=c=1$)

and chemical potential μ

neglecting the particle mass (okay since in interesting region $E = 3T \gg m$)

and chemical potential (good as long as no additive quantum number)

Boson number density $n = \int N(E) d^3p = \frac{4\pi g}{(2\pi)^3} \int \frac{p^2 dp}{\exp\left(\frac{p}{T}\right) - 1}$

$$n = \frac{g}{\pi^2} T^3 \zeta(3)$$

with Riemann ζ -function $\zeta(3) \approx 1.2$

Thermodynamics of a relativistic Bose gas

Boson energy density $\epsilon = \int N(E) p d^3 p = \frac{4\pi g}{(2\pi)^3} \int \frac{p^3 dp}{\exp(\frac{p}{T}) - 1}$

$$\epsilon = \frac{3g}{\pi^2} T^4 \zeta(4)$$

with Rieman ζ -function $\zeta(4) = \frac{\pi^4}{90} \approx 1.08$

$$\epsilon = \frac{\pi^2}{30} g T^4$$

and we get the Energy per particle $\epsilon/n = 3T \frac{\zeta(4)}{\zeta(3)} \approx 2.7 T$

Boson pressure $P = n^2 \partial \frac{\epsilon}{n} / \partial n \rightarrow P = \frac{1}{3} \epsilon$

Entropy density $d\sigma = d\epsilon/T$ and $d\epsilon = \text{const.} T^3 dT$

$$\sigma = \int d\sigma = \text{const.} \int T^2 dT = \frac{1}{3} \text{const.} T^3$$

$$\sigma = \frac{4\pi^2}{90} g T^3$$

$$\rightarrow \sigma = \frac{1}{3} \frac{d\epsilon}{dT}$$

Thermodynamics of a relativistic Bose gas

and the entropy per particle (boson) $\sigma/n = 4\zeta(4)/\zeta(3) \approx 3.6$

cf. old Landau formula for pions: $S = 3.6 dN/dy$

3.1.2 Thermodynamics of a relativistic Fermi gas

probability density for occupation $N(E) = \frac{g}{(2\pi)^3} (\exp(\frac{E - \mu}{T}) + 1)^{-1}$

but now μ not generally 0

Fermion number density

$$n = \frac{4\pi g}{(2\pi)^3} \int \frac{p^2 dp}{\exp(\frac{p - \mu}{T}) + 1}$$

Energy density

$$\epsilon = \frac{4\pi g}{(2\pi)^3} \int \frac{p^3 dp}{\exp(\frac{p - \mu}{T}) + 1}$$

} cannot be solved
analytically, only
numerically

but there is analytic solution for sum of particle and antiparticle (e.g. quark and antiquark)
(Chin, PLB 78 (1978) 552)

and

$$\epsilon_q + \epsilon_{\bar{q}} = g \left(\frac{7\pi^2}{120} T^4 + \frac{\mu^2}{4} T^2 + \frac{\mu^4}{8\pi^2} \right)$$

$$n_q - n_{\bar{q}} = g \left(\frac{\mu}{6} T^2 + \frac{\mu^3}{6\pi^2} \right)$$

specific example for fermions: quarks in QGP with no net baryon density

$$\langle q \rangle = \langle \bar{q} \rangle \leftrightarrow \mu = 0$$

in that case quark number density $n_q = \frac{g}{\pi^2} T^3 d(3)$

Note: $d(\alpha + 2) = \int \frac{x^\alpha dx}{e^x + 1}$ and $d(3) \approx 0.9$

and quark and antiquark energy density $\epsilon_q = \epsilon_{\bar{q}} = \frac{3g}{\pi^2} T^4 d(4) = \frac{7\pi^2}{240} g T^4$

$$\text{with } d(4) = \frac{7\pi^4}{720}$$

the energy per quark is then $\epsilon/n = 3T \frac{d(4)}{d(3)} \approx 3.2T$

Entropy density (computed as above for bosons)

$$\sigma = \frac{7\pi^2}{180} g T^3$$

and the entropy per fermion (quark) $\sigma/n = 4 \frac{d(4)}{d(3)} \approx 4.2$

Summary relativistic bosons and fermions (no chem.pot.)

- Energy density $\epsilon \propto T^4$
- Pressure $P = \frac{1}{3}\epsilon \propto T^4$
- Entropy density $\sigma \propto T^3$
- Particle number density $n \propto T^3$
- ➔ to obtain physical units of GeV/fm³ or fm⁻³, multiply with appropriate powers of $\hbar c$
- all are proportional to the number of degrees of freedom
- between bosons and fermions there is a factor 7/8

$$\epsilon_f = \frac{7}{8}\epsilon_b \quad \text{etc.}$$

3.1.3 Short excursion: the bag model

to deal with QCD in the nonperturbative regime (i.e. where α_s is not negligible) one needs to make models (alternative: lattice QCD see below) for instance to treat the nucleon and its excitations

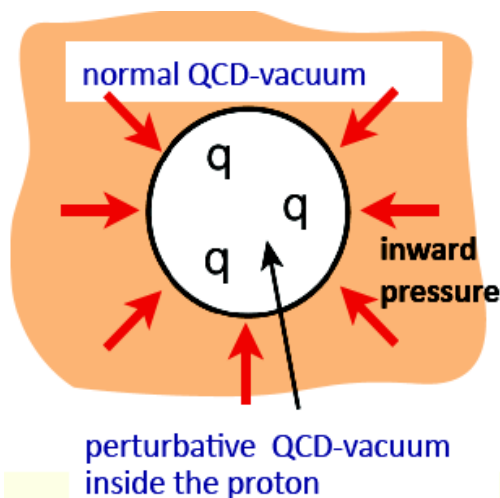
MIT bag model: build confinement and asymptotic freedom into simple phenomenological model

A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorne, Phys. Rev. D10 (1974) 2599

T. DeGrand, R.L. Jaffe, K. Johnson, J. Kiskis, Phys. Rev. D12 (1975) 2060

hadrons considered as bags embedded into a non-perturbative QCD vacuum also called “physical vacuum” or “normal QCD vacuum”

space divided into 2 regions



Interior of bag: quarks have very small (current) masses, interaction weak

Exterior of bag: quarks are not allowed to propagate there, lower vacuum energy, no colored quarks or gluons but quark and gluon condensates

Hadrons in MIT bag model

Hadrons are considered drops of another, perturbative phase of QCD immersed into normal QCD vacuum

all non-perturbative physics included in one universal quantity, the bag constant B defined as the difference in energy density between perturbative and physical vacua:

$$\epsilon_{\text{bag}} - \epsilon_{\text{vac}} \equiv B > 0$$

solve Dirac equation for massless quarks inside bag with volume V and surface S with special boundary conditions at the surface that

- i) enforce confinement: quark current normal to bag surface = 0
- ii) define a stability condition for bag: pressure of Dirac particles inside is balanced by difference in energy density inside and outside

$$H = H_{\text{kin}} + H_{\text{spin-spin}} + B V$$

The diagram shows the equation $H = H_{\text{kin}} + H_{\text{spin-spin}} + B V$ at the top. Three arrows point downwards from the terms to their respective descriptions:

- An arrow from H_{kin} points to "kin. Energy of quarks confined in bag".
- An arrow from $H_{\text{spin-spin}}$ points to "spin-spin interaction".
- An arrow from $B V$ points to "energy to make hole of volume V in phys. vacuum".

Hadrons in MIT bag model

for (nearly) massless quarks $E_{\text{kin}} \propto 1/R$ \longrightarrow tries to extend bag
(spherical bag with radius R)
bag term $B \frac{4\pi}{3} R^3$ \longrightarrow tries to contract bag
 \implies equilibrium is reached

obtain e.g. for nucleon mass (spherical bag with 3 quarks in s-state)

$$E = 3 \frac{\omega_{n,-1}}{R} + \frac{4\pi}{3} B R^3 \quad \text{with} \quad \omega_{1,-1} = 2.04 \quad \omega_{2,-1} = 5.40 \quad \dots$$

and $\frac{\partial E}{\partial R} = 0$

internal energy determines the radius of the bag, if B is a universal constant

- determines masses and sizes of all hadrons
- rather successful with $B_{\text{MIT}} = 56 \text{ MeV}/\text{fm}^3$
baryon octet and decuplet as well as vector mesons well reproduced

note: often instead of B, $B^{1/4}$ in MeV is quoted $B_{\text{MIT}}^{1/4} = 146 \text{ MeV}$

3.1.4 Thermodynamics of pion gas and QGP

pion gas: massless bosons with degeneracy $g_\pi = 3$ for π^+, π^0, π^-

energy density of pion gas $\epsilon_\pi = \frac{\pi^2}{30} g_\pi T^4 = 129 T^4$ and pressure $P = \frac{1}{3} \epsilon = 43 T^4$

after properly inserting missing powers of $\hbar c$ and using T in GeV

quark-gluon plasma:

gluons as massless bosons with degeneracy $g_g = 2(\text{spin}) \times 8(\text{color}) = 16$

quarks massless fermions with degeneracy $g_q = N_f \times 2(\text{spin}) \times 3(\text{color}) = 6 N_f$

and same for antiquarks (here N_f is number of massless/light flavors)

additional contribution to energy density: to make quark-gluon gas, need to create cavity in vacuum

energy needed is given by the **bag constant B** “pressure of vacuum on color field”

analogy to Meissner effect: superconductor expels magnetic field

↔ QCD vacuum expels color field into bags

$$\rightarrow \epsilon = \epsilon_{\text{thermal}} + B$$

and deriving pressure as above

$$P = \frac{1}{3}(\epsilon - 4B)$$

3.1.4 Thermodynamics of pion gas and QGP

What value to use for the bag constant?

from hadron phenomenology at $T=0$ and normal nuclear matter density

$$B \approx 50 - 100 \text{ MeV/fm}^3$$

but there are a number of problems with MIT bag model

and there is good indication that B derived there is not the energy density of the QCD vacuum; conclusion: hadrons are not small drops of the new QCD phase but only a relatively small perturbation of the QCD vacuum

also $B = B(T, n)$ (see e.g. Shuryak, the QCD vacuum...)

basic argument: at large T, n all non-perturbative phenomena suppressed

$$B_{\text{eff}} \approx 500 - 1000 \text{ MeV/fm}^3 \quad \text{vacuum energy density}$$

energy density of quark-gluon gas

$$\epsilon_{\text{qg}} = \frac{\pi^2}{30} (g_g + \frac{7}{8} g_q) T^4 + B = \frac{\pi^2}{30} (16 + \frac{21}{2} N_f) T^4 + B$$

for $N_f = 2$ (u,d)

$$\epsilon_{\text{qg}} = 1592 T^4 + 0.5 \left(\frac{\text{GeV}}{\text{fm}^3} \right)$$

Constructing the phase diagram

system always in phase with highest pressure

Gibbs conditions for critical point

$$P_{\text{QGP}} = P_{\text{piongas}}$$

and

$$\mu_{\text{QGP}} = \mu_{\text{piongas}} (= 0)$$

for $N_f = 2$
$$\frac{3\pi^2}{90} T_c^4 = \frac{\pi^2}{90} (16 + \frac{21}{2} N_f) T_c^4 - B$$

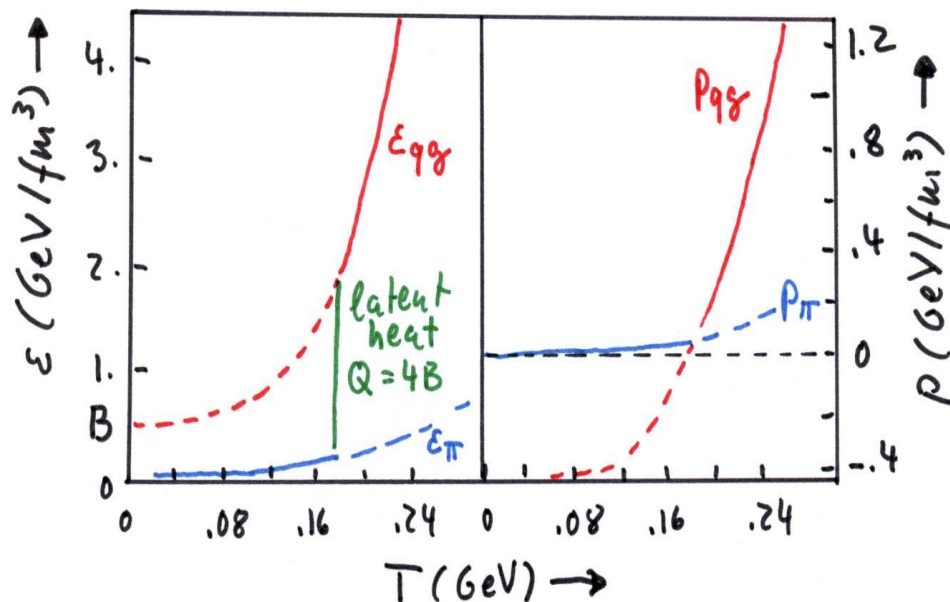
$$\frac{34\pi^2}{90} T_c^4 = B \quad T_c = \left(\frac{90 \cdot 0.5 \text{ GeV} \cdot 0.197^3 \text{ GeV}^3 \text{ fm}^3}{34\pi^2 \text{ fm}^3} \right)^{1/4} = 0.18 \text{ GeV}$$

latent heat:

$$\begin{aligned} \epsilon_{\text{qg}} - \epsilon_{\text{pion}} (\text{at } T_c) &= \frac{34\pi^2}{30} T_c^4 + B \\ &= 1.54 + 0.5 = 2 \frac{\text{GeV}}{\text{fm}^3} \end{aligned}$$

change in entropy density:

$$\sigma_{\text{qg}} - \sigma_{\text{pion}} = \frac{34 \cdot 4\pi^2}{90} T_c^3 = 11.4 / \text{fm}^3$$

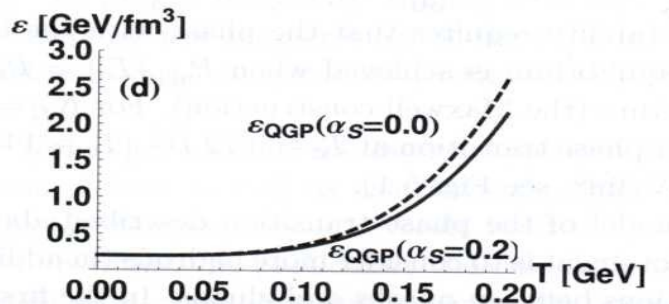
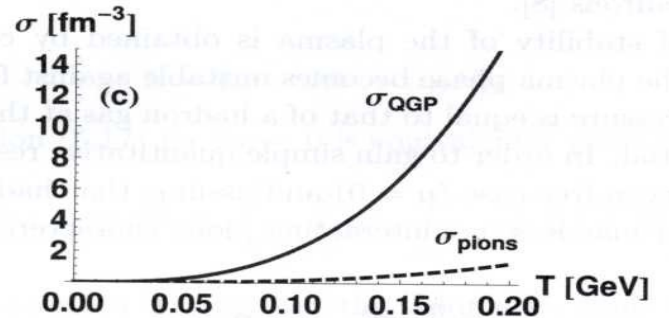
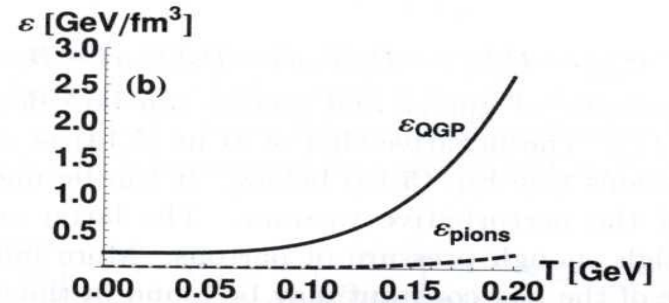
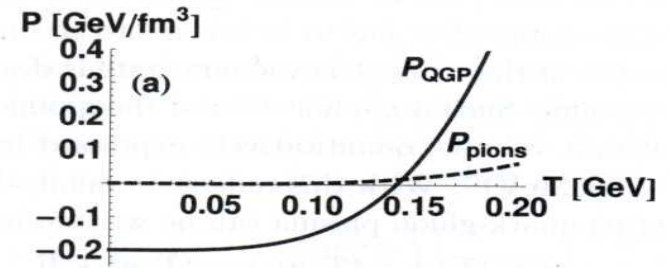


summary pion gas and QGP at zero baryon chemical potential

pressure
energy density
entropy density
for a gas of massless pions and an ideal
quark-gluon plasma with 2 quark flavors

(note: here $B^{1/4}=200$ MeV or
 $B=0.209$ GeV/fm³ is used)

effect of finite coupling constant in QGP
reduces energy density slightly
- first order perturbation theory
S.A.Chin Phys. Lett. B78 (1978) 552



Now check the high baryon density limit

compute a $T = 0$ $\mu \neq 0$ point
cannot do this with pions alone, need nucleons

$$P_{\text{pion}} = 0 \quad P_{\text{nucleon}} = \frac{g\mu^4}{3 \cdot 8\pi^2} \quad \text{with } g=4 \text{ (2(spin) x 2(isospin))}$$

for the quark-gluon side at $T=0$

$$P_{q\bar{q}} = \frac{g\mu^4}{3 \cdot 8\pi^2} - B \quad \text{with } g=12 \text{ (quarks and antiquarks)}$$

$$P_{\text{nucleon}} = P_{q\bar{q}} \rightarrow$$

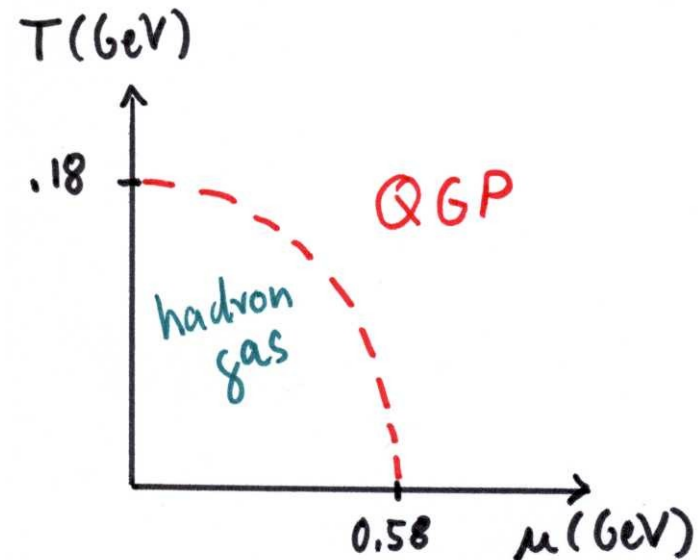
$$\mu = \left(\frac{3\pi^2 \cdot 0.5 \text{ GeV} \cdot 0.197^3 \text{ GeV}^3 \text{ fm}^3}{\text{fm}^3} \right)^{1/4}$$

$$= 0.58 \text{ GeV}$$

simple thermodynamik model gives
first order phase transition,

Caution: this sets the scale, but there are
a number of approximations

- pion gas is oversimplification for hadronic matter
- $B = 0.5 \text{ GeV}/\text{fm}^3$ (should use $B(T, n)$)



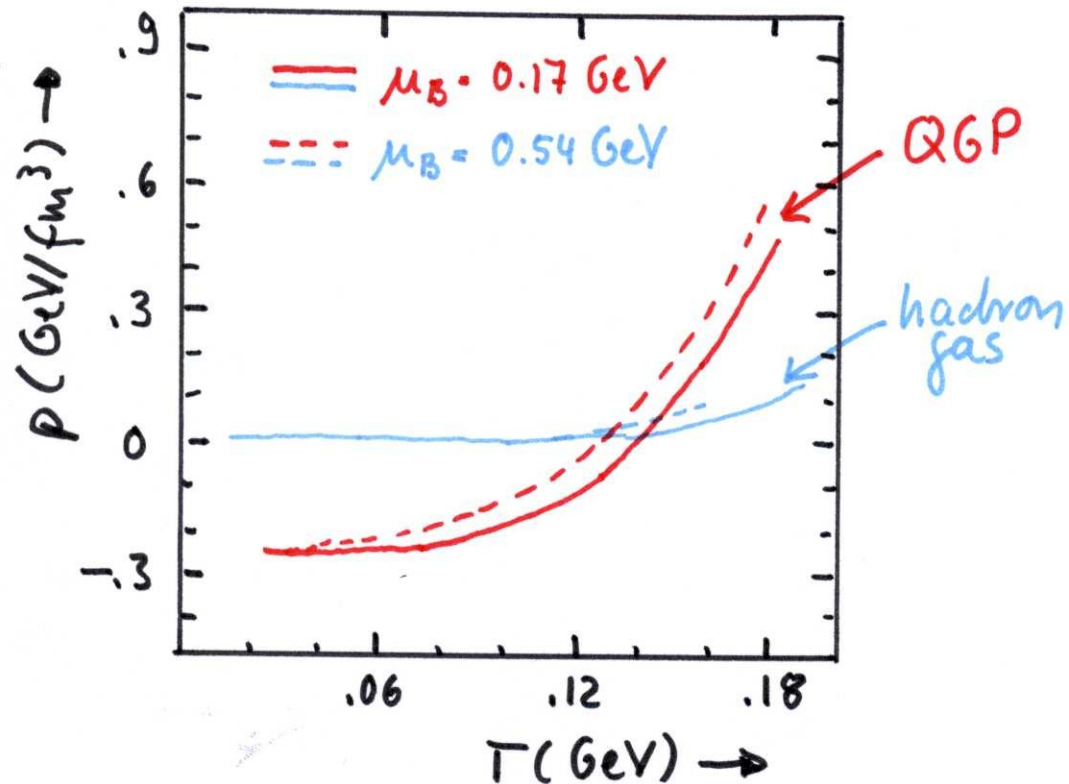
3.1.5 more realistic: replace pion gas by hadron gas

implement all known hadrons up to 2 GeV in mass

ideal gas of quarks and gluons, u, d massless, s 150 MeV

fix bag constant to match lattice QCD result (see below) at $\mu_b=0 \rightarrow B=262 \text{ MeV/fm}^3$

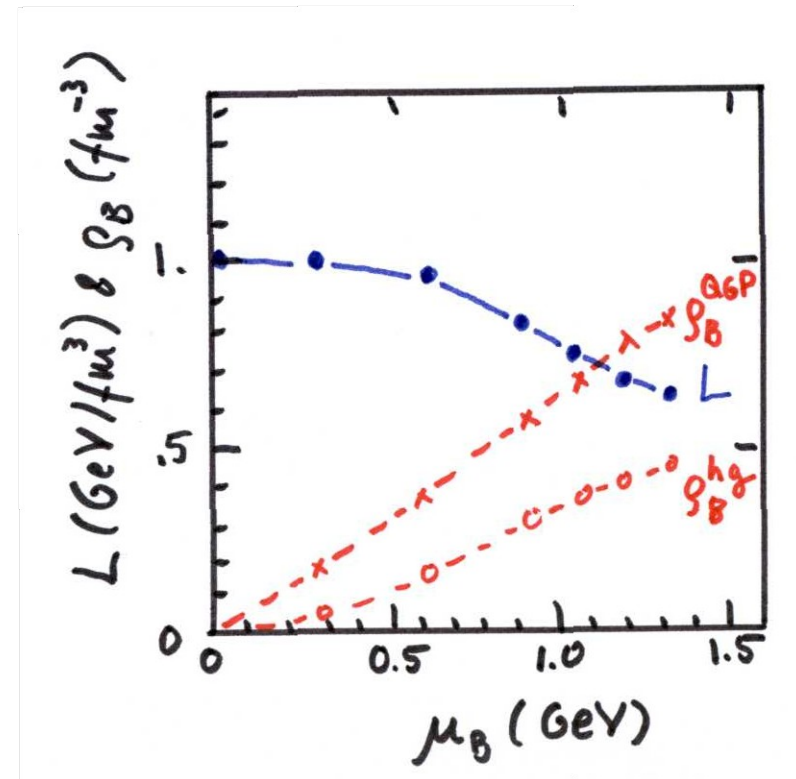
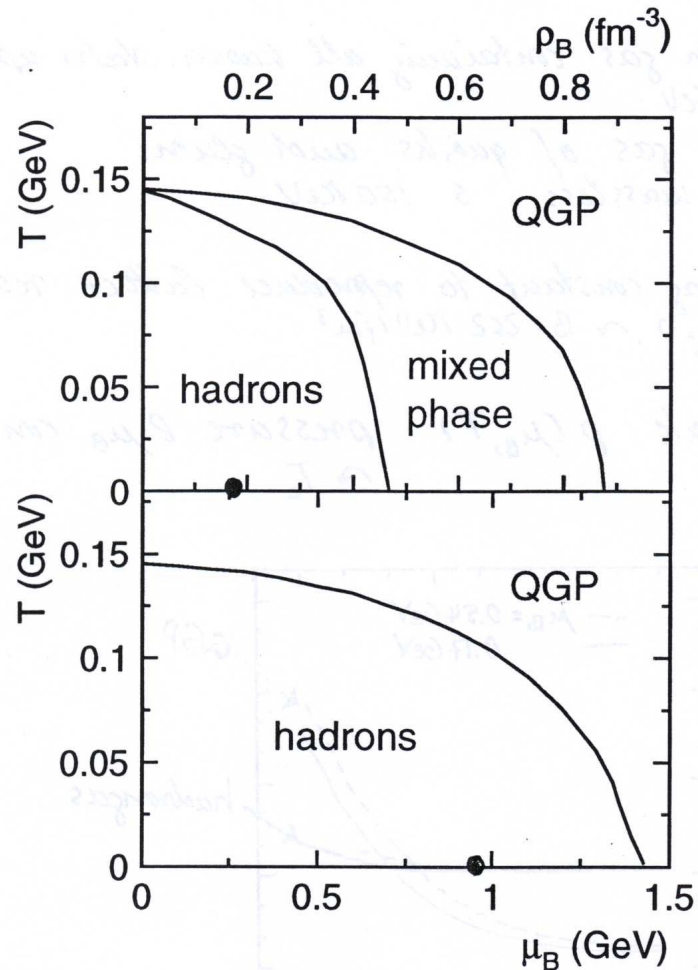
compute $P(\mu_b, T)$ with
with pressure and μ_b continuous
to obtain T_c



P. Braun-Munzinger, J. Stachel Nucl.Phys. A606 (1996) 320

Phase diagram constructed with hadron gas and QGP

P. Braun-Munzinger, J. Stachel Nucl.Phys. A606 (1996) 320



Note: chemical potential is continuous at phase transition but not the baryon density!

Speed of sound

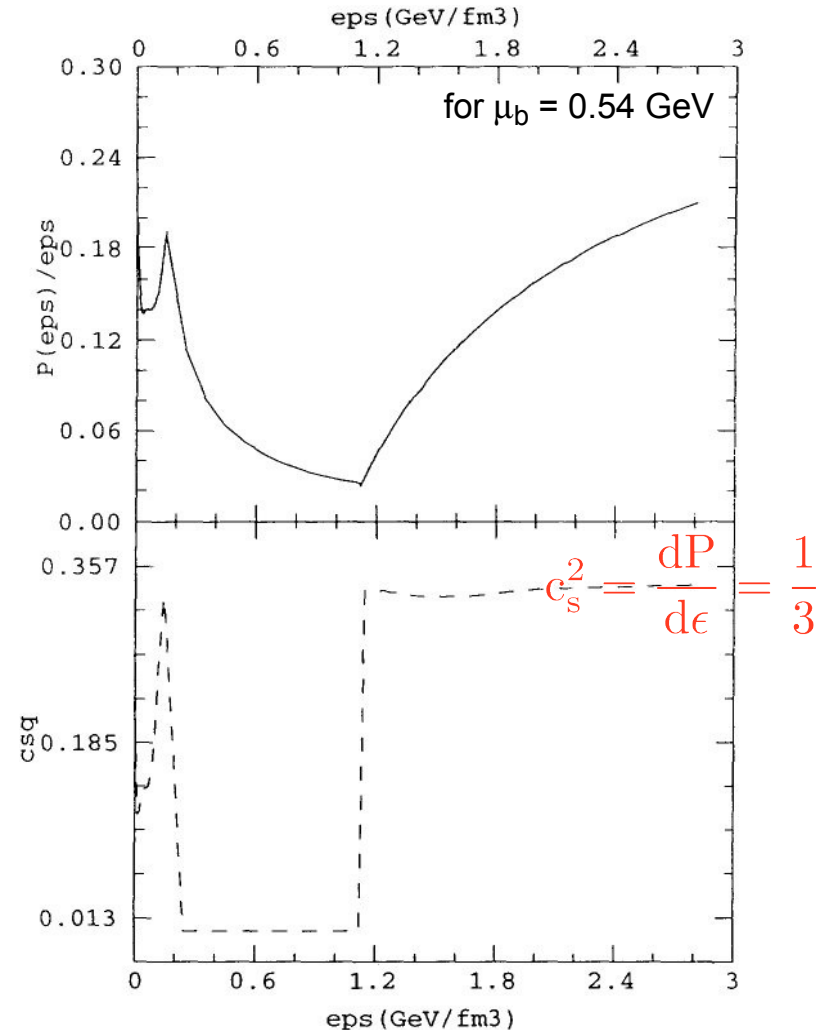
in relativistic gas, speed of sound squared

$$c_s^2 = \frac{dP}{d\epsilon} = \frac{1}{3} \quad \text{without interactions}$$

but in vicinity of phase transition strong deviation of P from $1/3 \epsilon$

there is always a minimum in speed of sound

leading to a so-called 'softest point'



3.2 Lattice QCD

QCD asymptotically free at large T and/or small distances
at low T and for finite size systems $\alpha_s = O(1)$

→ cannot use perturbation theory

instead solve QCD numerically at zero and finite temperature by putting gauge field on a space-time lattice \longleftrightarrow “lattice QCD”

some references: M. Creutz 'Quarks, Gluons and Lattices (Cambridge U. Press, Cambridge, 1983)
J. Negele, Proc. NATO Advanced Study Institute, 'Hadrons and Hadronic Matter", eds. D. Vautherin et al. (Plenum, 1990)
Regular proceedings of Lattice conferences
Proc. Lattice 91, Nucl. Phys. B, Proc. Suppl.
“ Lattice 92, “
etc.
recent review: A. Ukawa arXiv: 1501.04215 [hep-lat], J.Stat.Phys. in print

Lattice QCD - schematic outline of basic (4) steps

i) use evolution in Euclidean time $\tau = it = 1/T$

to filter out hadronic ground state and/or evaluate thermal average

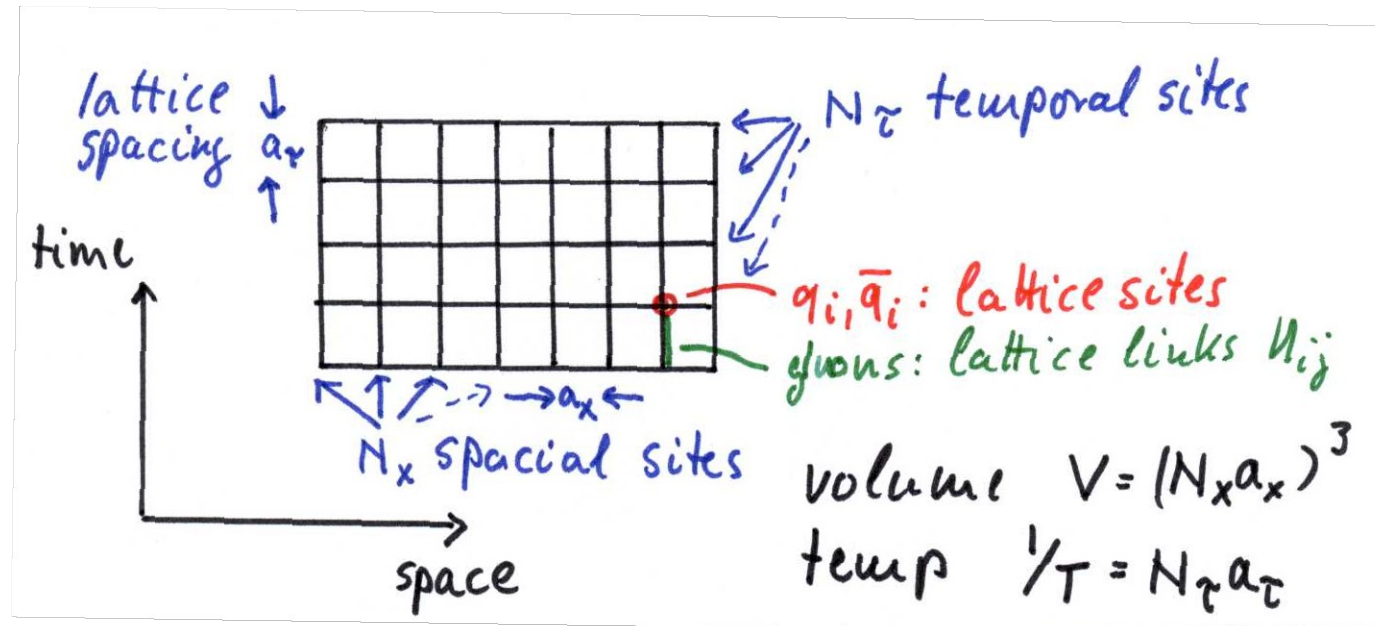
ii) replace Euclidean x, τ continuum by finite lattice

field theory with infinite number of degrees of freedom \rightarrow finite many body problem

quantum field theory equivalent to classical statistical mechanics

with

$$\exp(-iHt) \rightarrow \exp(-H\tau) = \exp(-S)$$



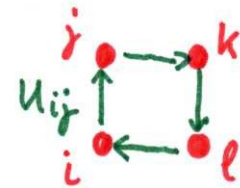
Lattice QCD basic steps

iii) evaluate partition function Z by using Feynman path integrals

$$Z = \text{Tr} \exp(-H_{\text{QCD}}\tau)$$

$$\longrightarrow Z = \int \prod_{\text{links}} dU_{ij} \exp(-S(U))$$

action, given by sum over elementary plaquette



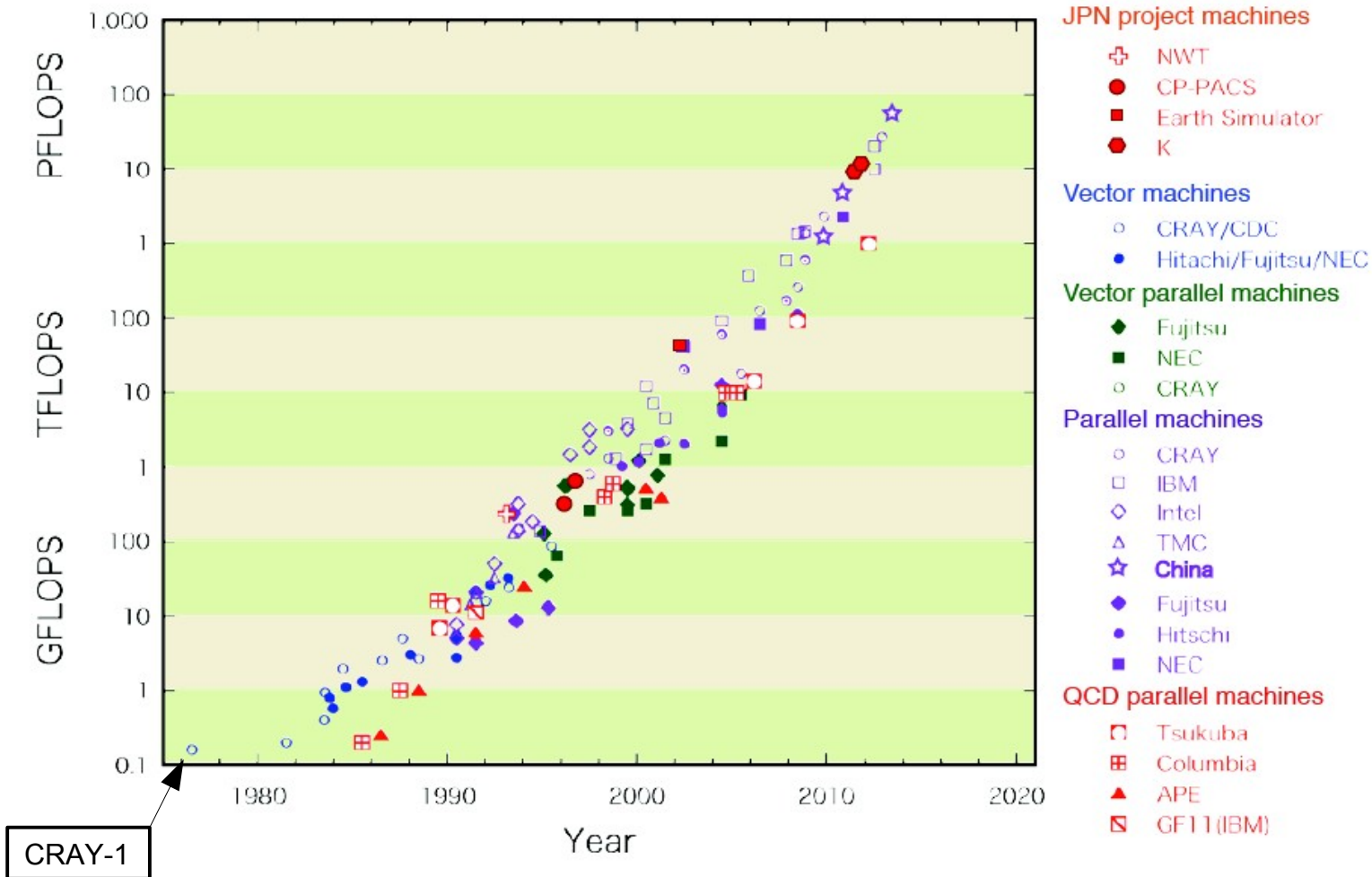
K. Wilson, Phys. Rev. D10 (1974) 2445

iv) lattices need to be big! e.g. $16^3 \times 32$ sites

have to sum over all color indices at each link \longrightarrow integral 10^7 dimensional
start with some values U_{ij} for all links, successively reassign new elements
to reduce computing time: use stochastic technique with clever weighting
($\exp(-S(U))$ favors small action)

have to sweep through entire lattice a few hundred times to evaluate thermodynamic quantities, baryon masses, wave functions

Huge increase in computing power since Wilson 1974

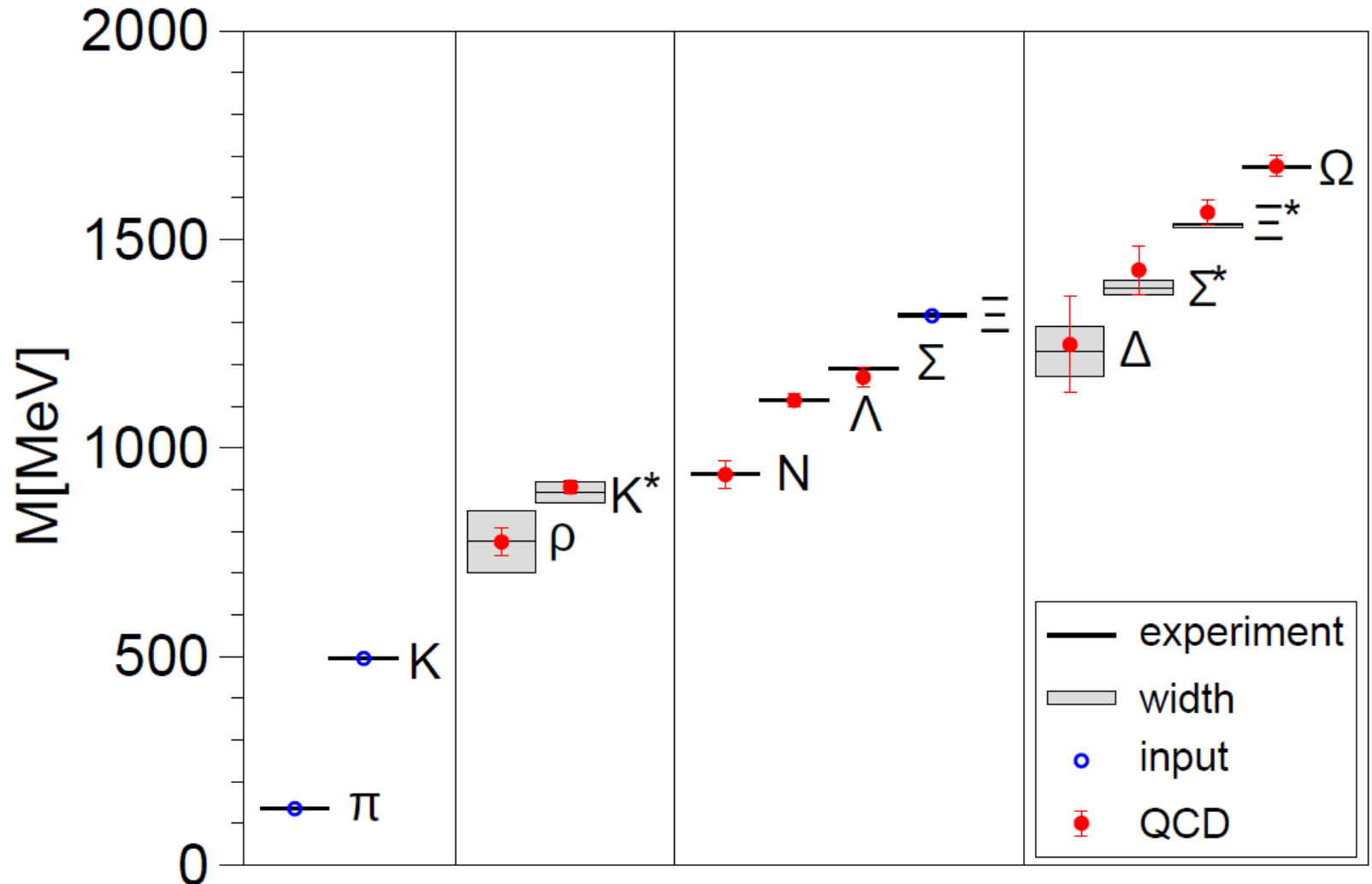


evolution of peak speed of supercomputers and improved algorithms:

—▶ now lattice QCD is a mature technique with increasingly reliable results

State-of-the-art light hadron spectrum from lattice QCD

S. Dürr, Z.Fodor et al. (Budapest-Marseille–Wuppertal Coll.) Science 322 (2008) 1225



Finite temperature lattice QCD

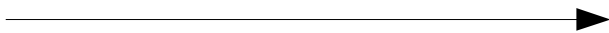
Variation of temperature:

temperature is changed e.g. by keeping Nt constant
and changing the lattice spacing

(and thereby the coupling g^2)

$$a \rightarrow 0, \quad g^2 \rightarrow 0, \quad T \rightarrow \infty$$

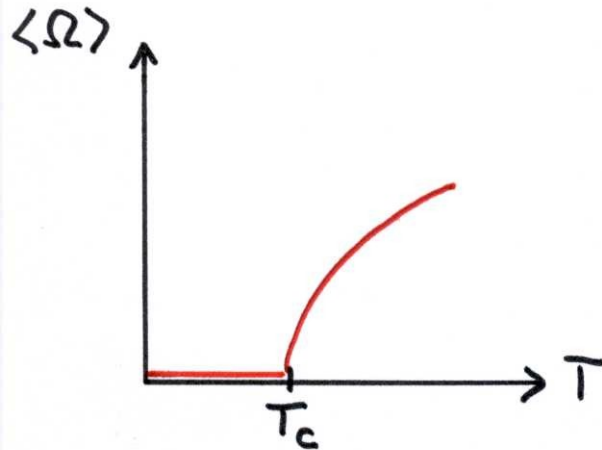
approach asymptotic freedom



absolute scale is set by
calculating a baryon or
meson mass
in units of a (or g^2)

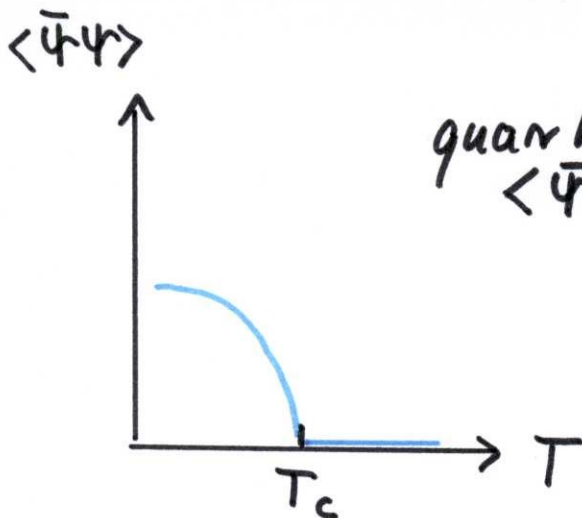
Indicators of the phase transition

Order parameter of deconfinement:



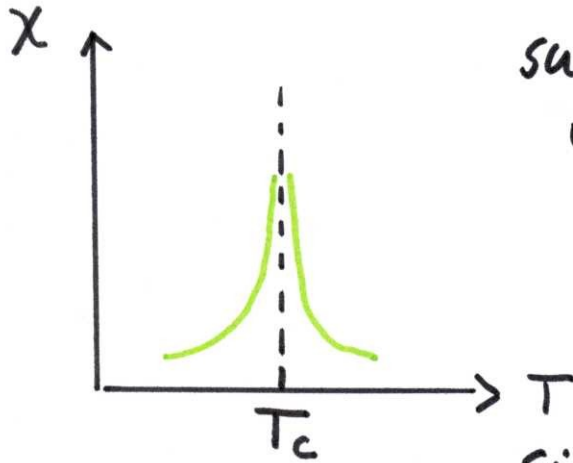
free energy of static quark
 $F_q = \infty$ confined
 $F_q < \infty$ deconfined
Wilson-Polyakov loop
 $\langle \Omega \rangle = \exp(-F_q/T) = 0$ confined
 $\neq 0$ deconfined

Order parameter of chiral symmetry restoration:



quark condensate
 $\langle \bar{\psi}\psi \rangle \neq 0$ broken chiral symm.
 $= 0$ chirally symm.

Indicators of the phase transition



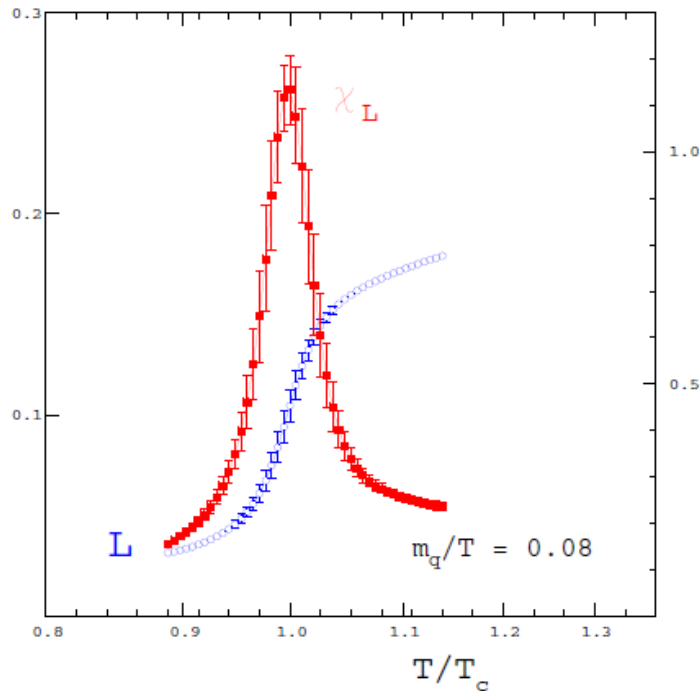
susceptibilities χ
e.g. chiral susc.
 $\chi_m = T \frac{\partial \langle \bar{\Psi} \Psi \rangle}{\partial m}$

response of quark condens.
to mass

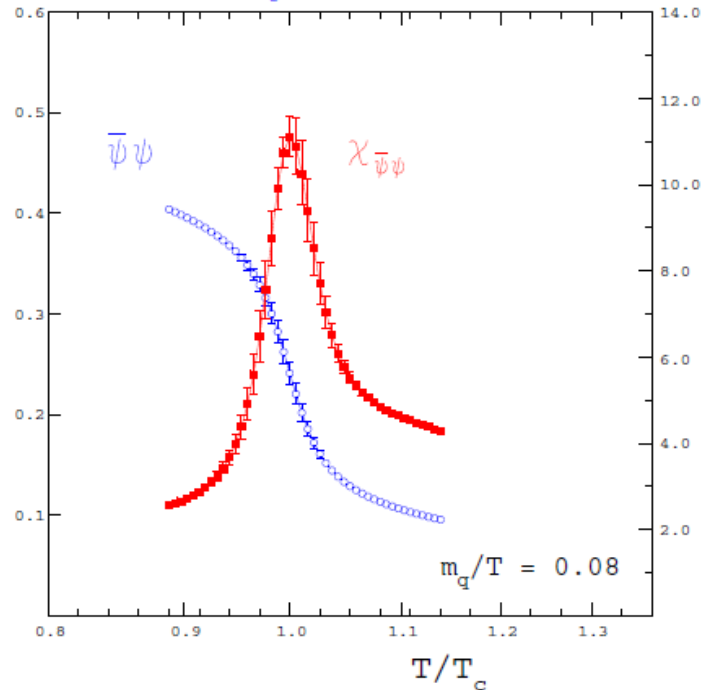
similarly Polyakov susceptibility
response to addition of
a quark

Deconfinement and chiral phase transition in lattice QCD

Polyakov loop - deconfinement



quark condensate $\langle\psi\psi\rangle = \partial p/\partial m$ -
chiral sym. restoration



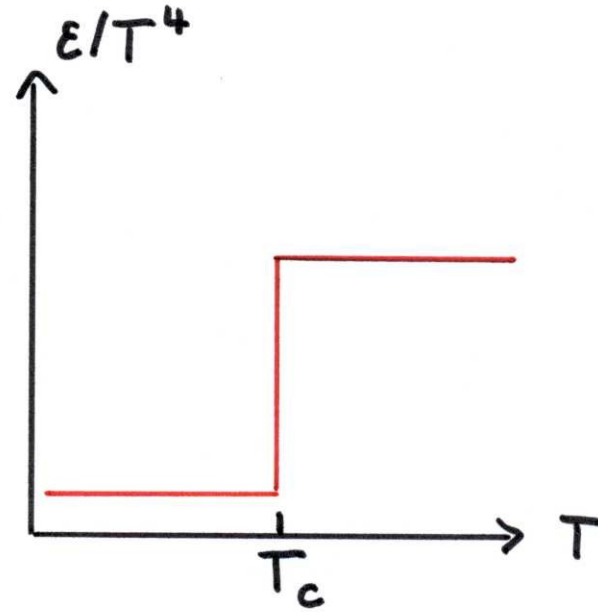
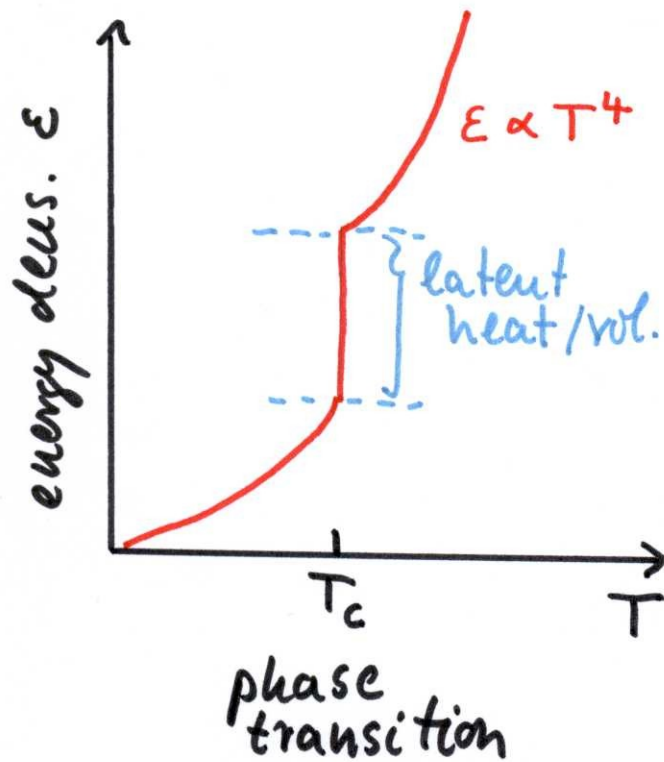
Suceptibilities χ : measure of fluctuations

$$\chi_L = N_\sigma^3 (\langle L^2 \rangle - \langle L \rangle^2)$$

$$\chi_{\psi\psi} = \partial \langle \psi\psi \rangle / \partial m = \partial^2 p / \partial m^2$$

F. Karsch, E. Laermann, hep-lat/0305025

How to display equation of state?



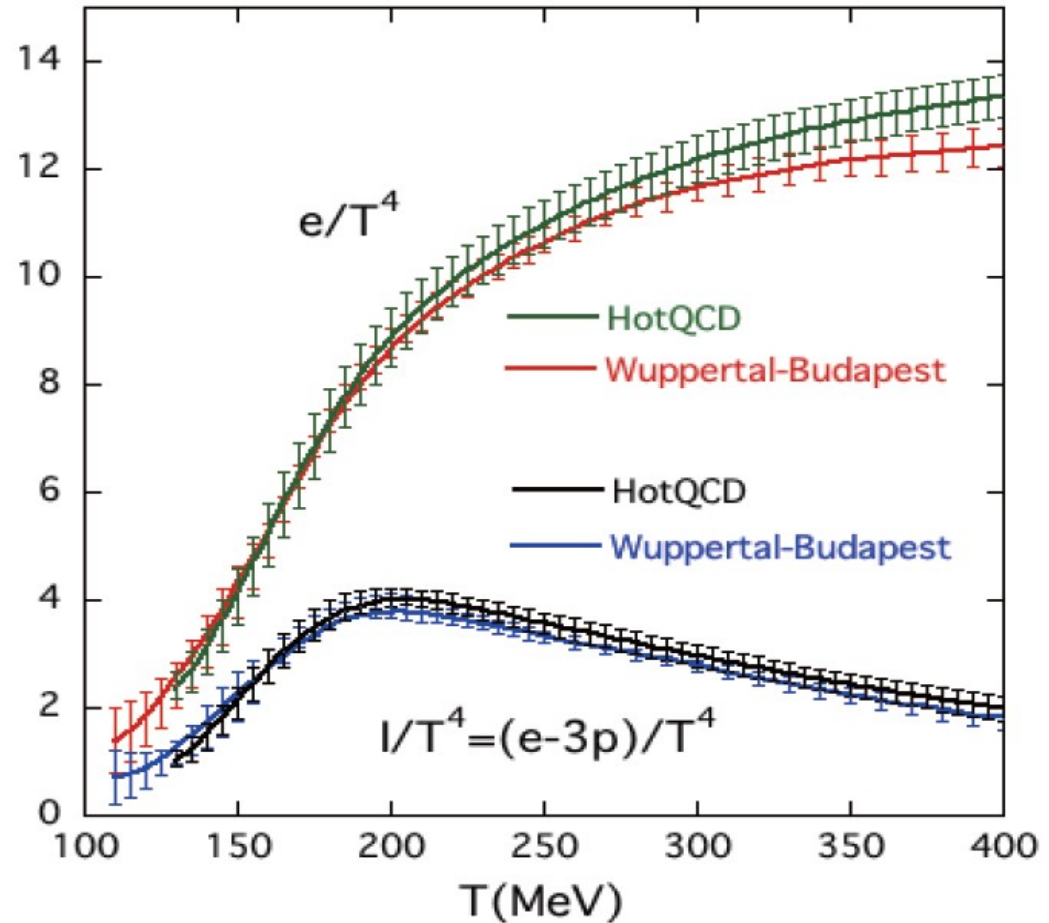
divide by T^4 dependence for relativistic Bose/Fermi-gas

Equation of state in lattice QCD

consolidated results from different groups, extrapolated to continuum and chiral limit

rapid rise of energy density (normalized to T^4 rise for relativistic gas)
- signals rapid increase in degrees of freedom due to transition from hadrons to quarks and gluons

A.Ukawa, arXiv:1501.04215

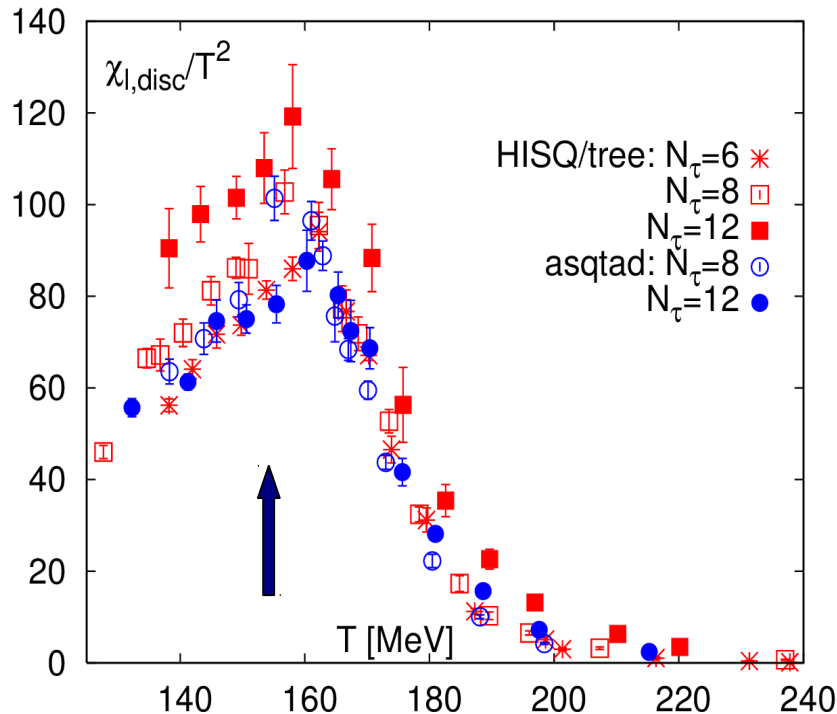


What is most realistic value of the critical temperature?

order parameter: chiral condensate, its susceptibility peaks at T_c

S.Borsanyi et al. Wuppertal-Budapest Coll., JHEP 1009 (2010) 073

A.Bazavov et al. HotQCD Coll., PRD 85 (2012) 054503



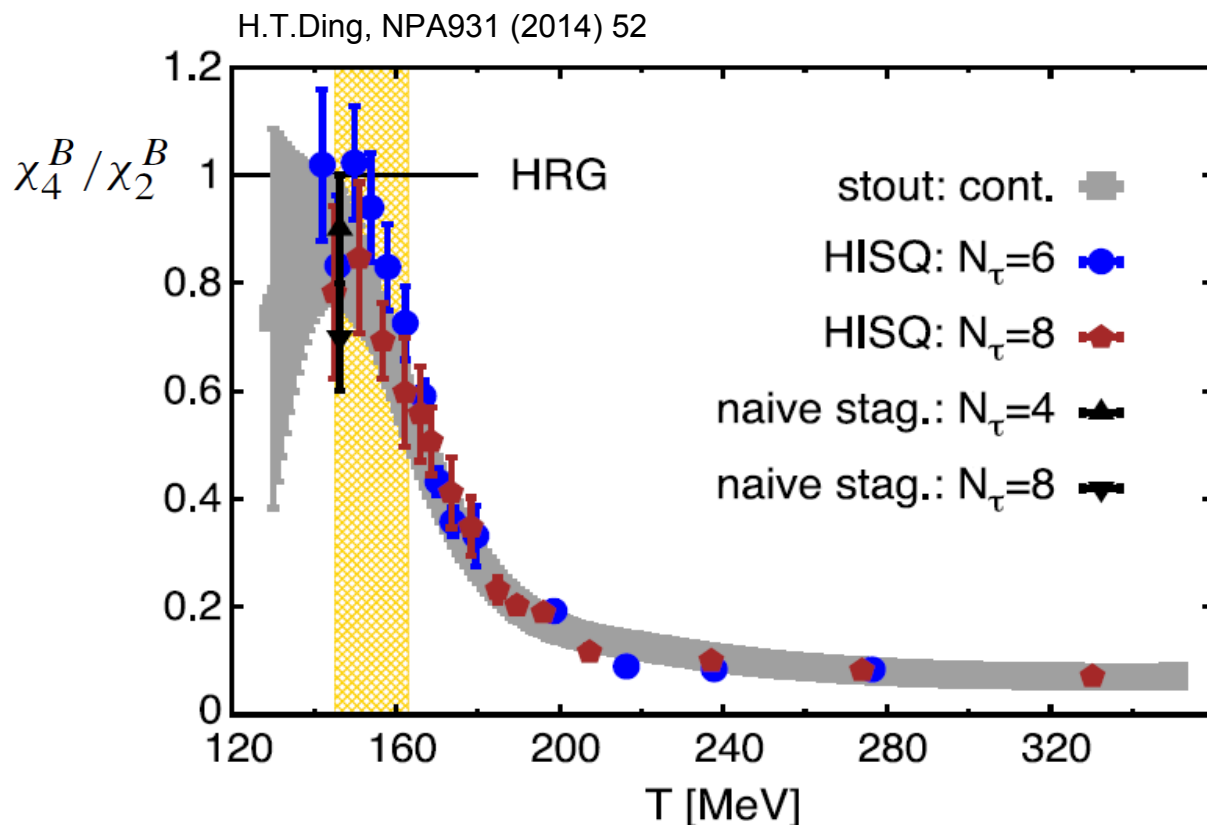
$$\langle \bar{\Psi} \Psi \rangle = \frac{T}{V} \frac{\partial \ln Z}{\partial m}$$

$$\chi_{\bar{\Psi} \Psi} = \frac{T}{V} \frac{\partial^2 \ln Z}{\partial m^2}$$

T_c from peak in chiral susceptibility
= 150 – 160 MeV for chiral restoration

Measure of deconfinement in IQCD

$$\chi_4^B / \chi_2^B \propto \text{baryon number}^2$$



← confined: 1

measure suggested by
Ejiri, Karsch, Redlich
(2006)

← deconfined: $6/9\pi^2$

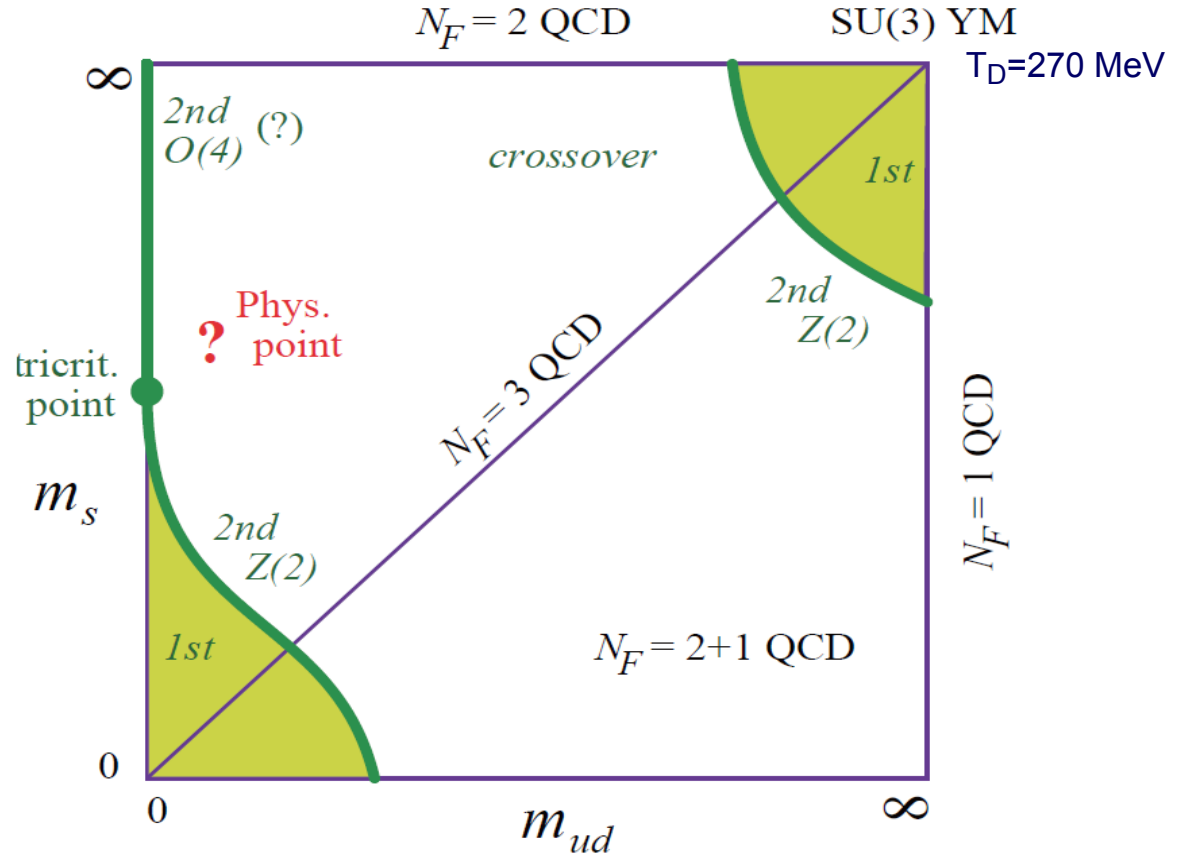
rapid drop suggests:

chiral cross over and deconfinement appear in the same narrow temperature range

Order of phase transition

present state-of-the art lattice QCD simulations give smooth cross over for realistic quark masses

critical role of strange quark mass



Phase diagram in 2+1 flavor QCD (Kanaya, 2010)

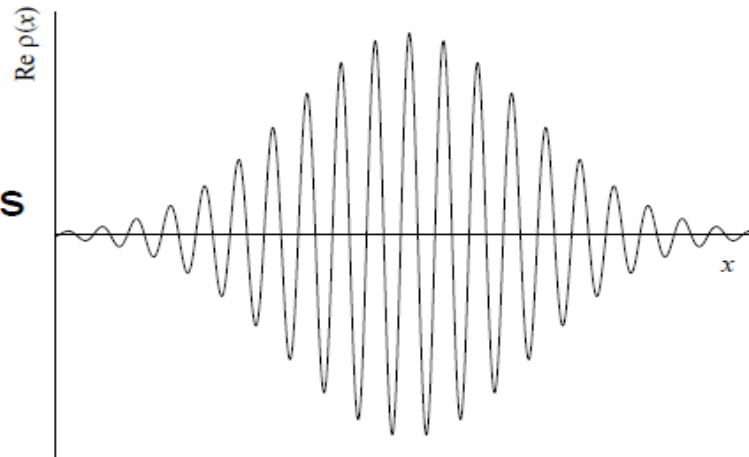
Lattice QCD at finite baryon density

Lattice QCD at non-equal numbers of fermions and antifermions (non-zero baryon chemical potential) has a problem:

- for Fermions the partition function contains a Slater determinant
- for non-zero chemical potential this Slater determinant is complex
- straight forward importance sampling not possible
- oscillations i.e. the lattice QCD sign problem

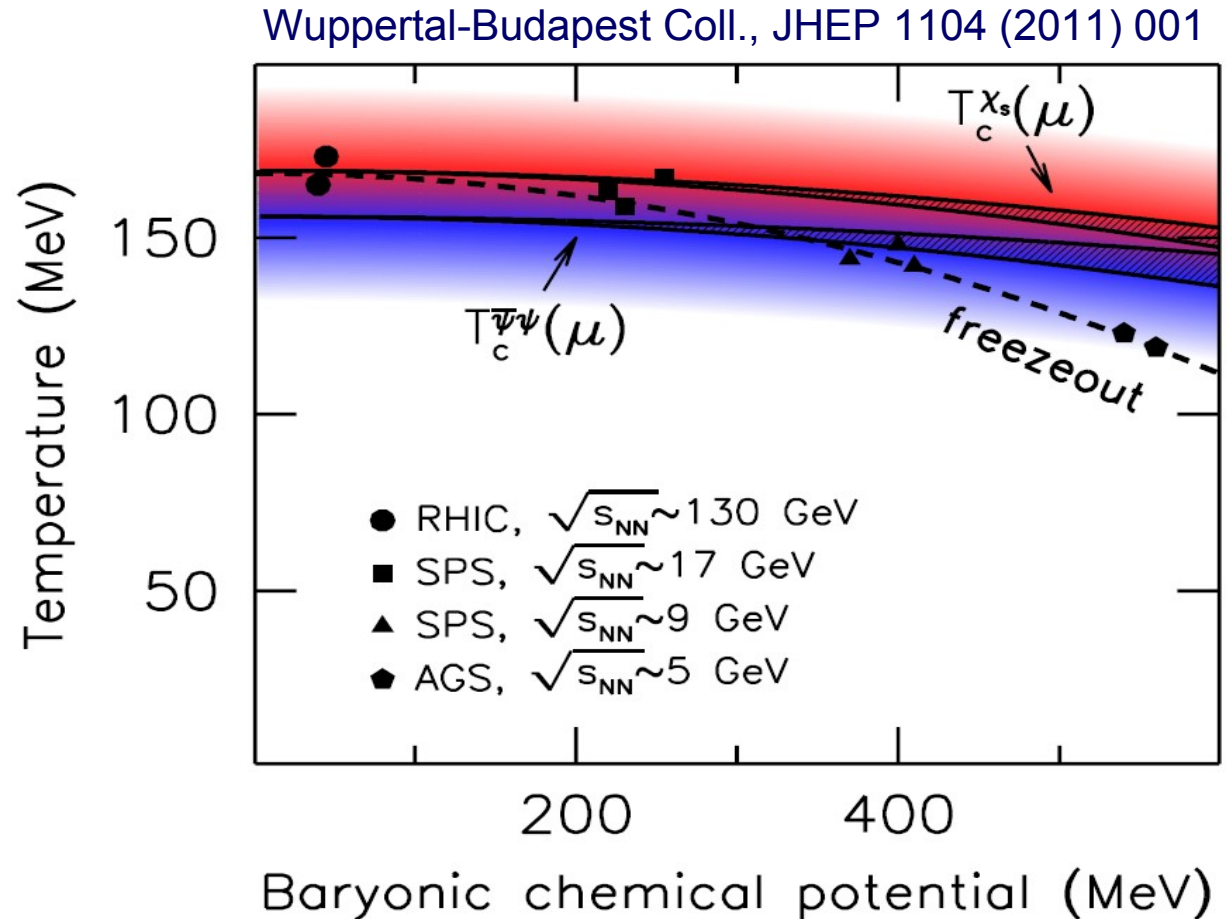
$$\det M(\mu) = |\det M(\mu)|e^{i\theta}$$

dominant configurations
in the path integral?



in last 10 years different methods have been developed to extrapolate into region of finite chemical potential

Lattice QCD at finite baryon density

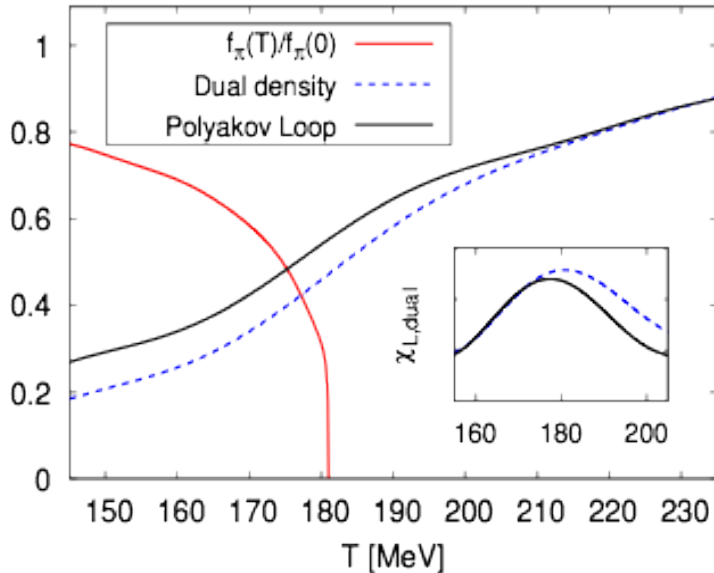


More reliable at finite baryon density: models of QCD

continuum QCD computation

Chiral limit

Braun, Haas, Marhauser, Pawłowski,
Phys.Rev.Lett.106:022002, 2011



first principle
determination

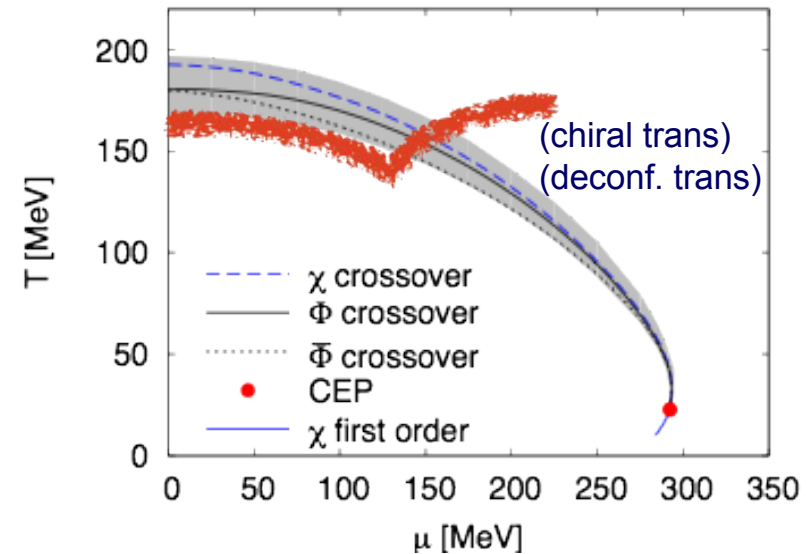


of model
parameters
(effective
potentials)

PNJL - QCD effective model

Physical quark masses

Herbst, Pawłowski, Schaefer, Phys.Lett. B696 (2011) 58



- in continuum QCD: closely related chiral & deconf. transition lines

- in PNJL follow these transition lines into phase diagram

- smooth crossover in region where calc already reliable $\frac{\mu_b}{T} < 2$ (above red line) no indication yet for critical endpoint

Speed of sound from lattice QCD

similar to what was visible already for hadron gas – QGP using bag model equation of state:

in region of phase transition P/ε not constant

softest point and minimum in speed of sound

