Quark-Gluon Plasma Physics

2. Kinematic Variables

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SS 2019

Lorentz transformation

Postulates

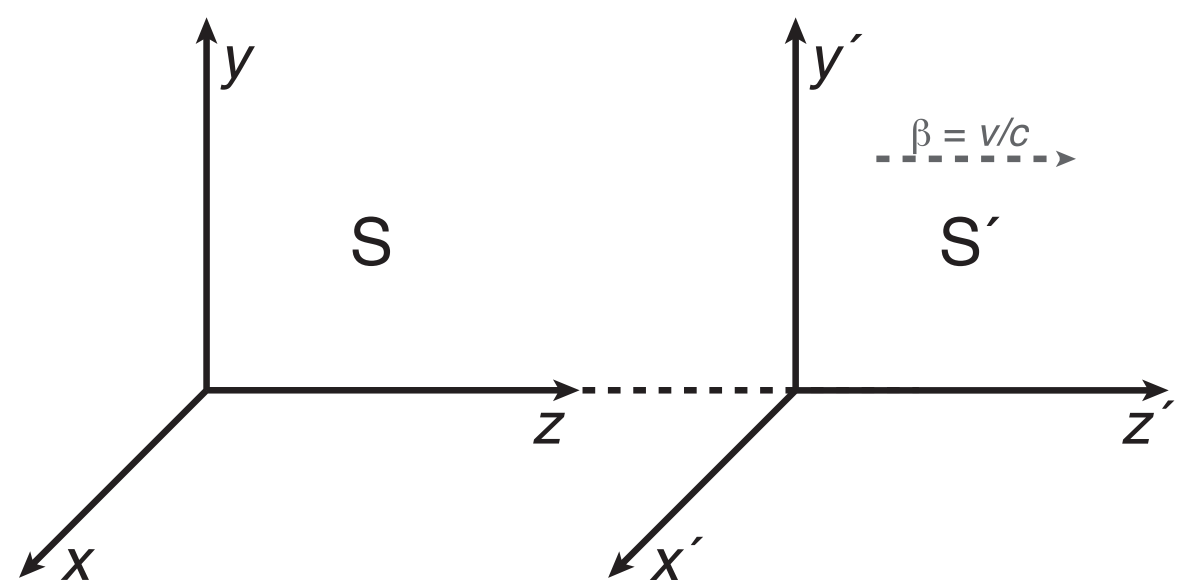
1. There is no preferred inertial frame
2. The speed of light in vacuum has the same value *c* in all inertial frames of reference

(Contravariant) space-time four-vector in system S:

$$x^\mu := (x^0, x^1, x^2, x^3) = (t, \vec{x}) = (t, x, y, z)$$

In system S'
(follows from the two postulates)

$$x^{0\prime} = \gamma(x^0 - \beta x^3)$$
$$x^{1\prime} = x^1$$
$$x^{2\prime} = x^2$$
$$x^{3\prime} = \gamma(x^3 - \beta x^0)$$



$$\beta = v/c \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Energy-momentum four-vector

General four-vector:
transforms under Lorentz transformation like the space-time four-vector

Relativistic energy and momentum:

$$E = \gamma m, \quad p = \gamma \beta m, \qquad m = \text{rest mass} \qquad (\hbar = c = 1)$$

Contravariant four-momentum vector:

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, \vec{p}) = (E, \vec{p}_T, p_z) = (E, p_x, p_y, p_z)$$

Covariant four-vector:

$$x^\mu := (x^0, x^1, x^2, x^3) \quad \rightarrow \quad x_\mu := (x^0, -x^1, -x^2, -x^3)$$

Scalar product of two four-vectors *a* and *b*:

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b}$$

Relation between energy and momentum:

$$E^2 = p^2 + m^2$$

Center-of-Mass System (CMS) [actually: center-of-momentum system]

Consider a collision of two particles. The CMS is defined by

$$\vec{p}_a = -\vec{p}_b$$

$$p_a = (E_a, \vec{p}_a) \qquad p_b = (E_b, \vec{p}_b)$$

The Mandelstam variable s is defined as

$$s := (p_a + p_b)^2 \overset{CMS}{=} (E_a + E_b)^2$$

$\sqrt{s}$ is the total energy in the center-of-mass frame ("center-of-mass energy")

Example: LHC. beam energy 6.5 TeV: $\sqrt{s} = 2\,E = 13$ TeV (lab frame = CMS)

More on LHC energies

From 'centripetal force = Lorentz force' one obtains:

$$R \equiv \frac{p}{q} = r_{\text{LHC,bend}} \cdot B_{\text{LHC}}, \qquad B_{\text{LHC,max}} \approx 8.3\,\text{T} \quad (\rightarrow \text{this limits } \sqrt{s})$$

"rigidity"   1232 dipoles x 14.3 m / (2 π) = 2804 m

protons: $R = p_{\text{proton}}$ ions: $R = \dfrac{A \cdot p_{\text{nucleon}}}{Z}$

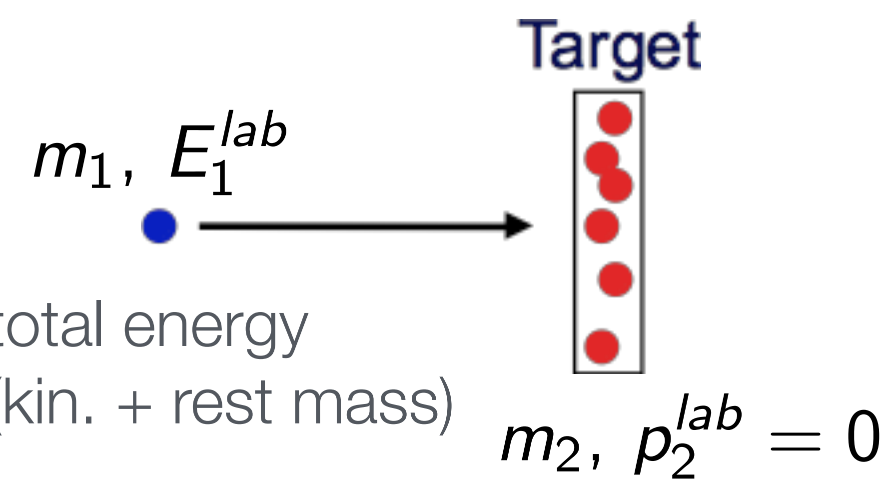
2011/12: $p_{\text{proton}} = 3.5\,\text{TeV} \rightarrow p_{\text{nucleon}} \equiv p_{\text{Pb}}/A = \dfrac{Z}{A} \cdot p_{\text{proton}} = 1.38\,\text{TeV}$

corresponding energy of nucleons in Pb ion for same *B* field (same rigidity)

Center-of-momentum energy per nucleon-nucleon pair:

Pb-Pb (2011/12): $\sqrt{s_{NN}} = 2.76\,\text{TeV}$ Pb-Pb (2015/18): $\sqrt{s_{NN}} = 5.02\,\text{TeV}$

$\sqrt{s}$ for Fixed-Target Experiments



$$s = \left[\begin{pmatrix} E_1^{\mathrm{lab}} \\ \vec{p_1} \end{pmatrix} + \begin{pmatrix} m_2 \\ \vec{0} \end{pmatrix} \right]^2$$

$$= m_1^2 + m_2^2 + 2E_1^{\mathrm{lab}} m_2$$

$$\Rightarrow \sqrt{s} = \sqrt{m_1^2 + m_2^2 + 2E_1^{\mathrm{lab}} m_2}$$

$$\overset{E_1^{\mathrm{lab}} \gg m_1,\, m_2}{\approx} \sqrt{2E_1^{\mathrm{lab}} m_2}$$

Example: antiproton production (fixed-target experiment):

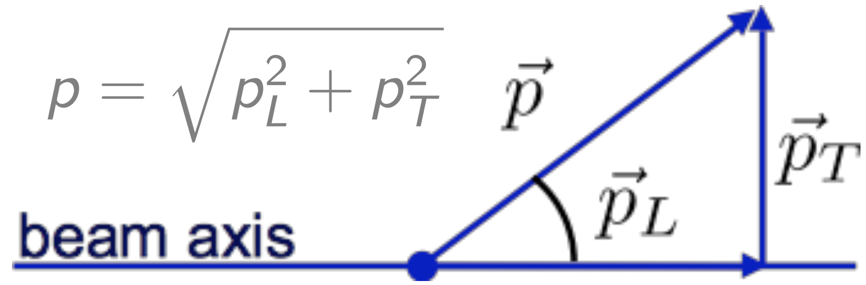
$$p + p \to p + p + p + \bar{p}$$

Minimum energy required to produce an antiproton: In CMS. all particles at rest after the reaction. i.e.. $\sqrt{s} = 4\, m_\mathrm{p}$. hence:

$$4m_p \overset{!}{=} \sqrt{2m_p^2 + 2E_1^{\mathrm{lab,min}} m_p} \quad \Rightarrow \quad E_1^{\mathrm{lab,min}} = \frac{(4m_p)^2 - 2m_p^2}{2m_p} = 7m_p$$

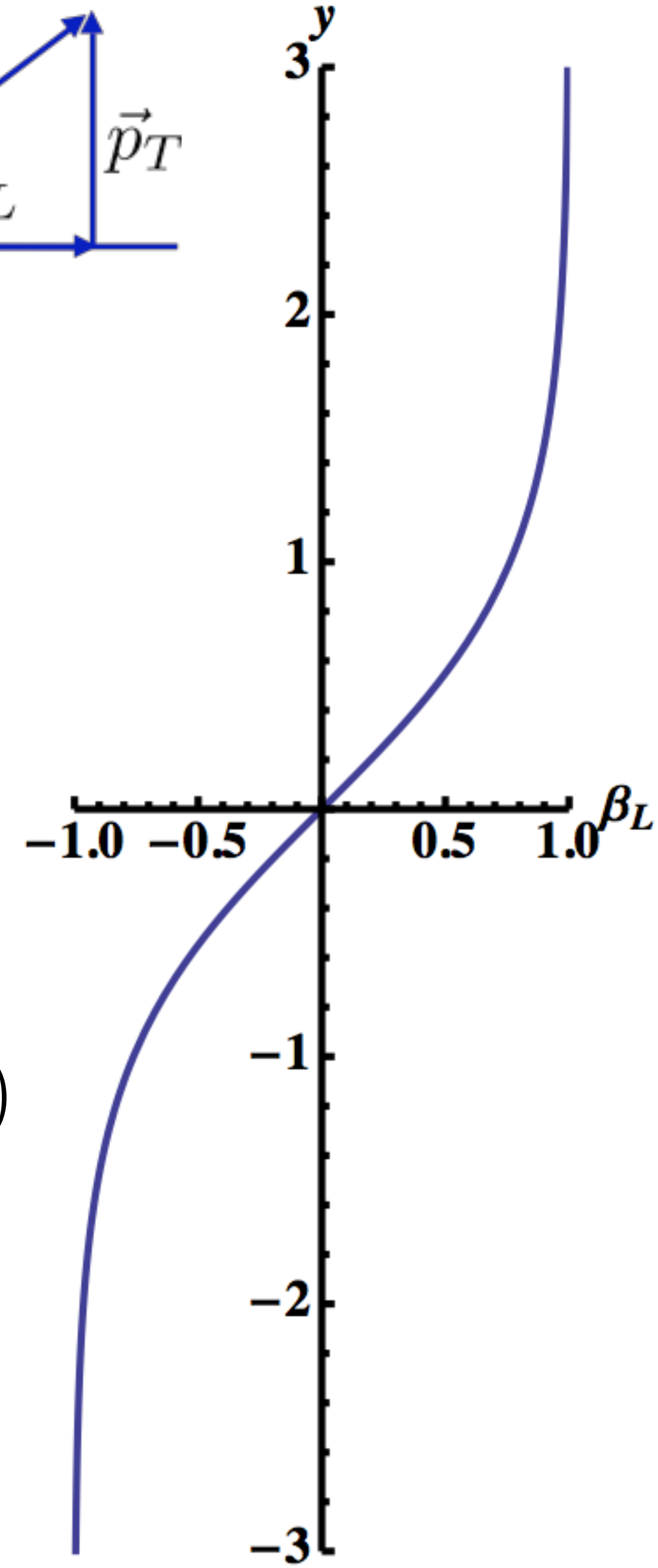
Rapidity

The rapidity y is a generalization of the (longitudinal) velocity $\beta_L = p_L / E$:



$$y := \operatorname{arctanh} \beta_L = \frac{1}{2} \ln \frac{1+\beta_L}{1-\beta_L} = \frac{1}{2} \ln \frac{E+p_L}{E-p_L}$$

$$y \approx \beta_L \quad \text{for} \quad \beta_L \ll 1$$



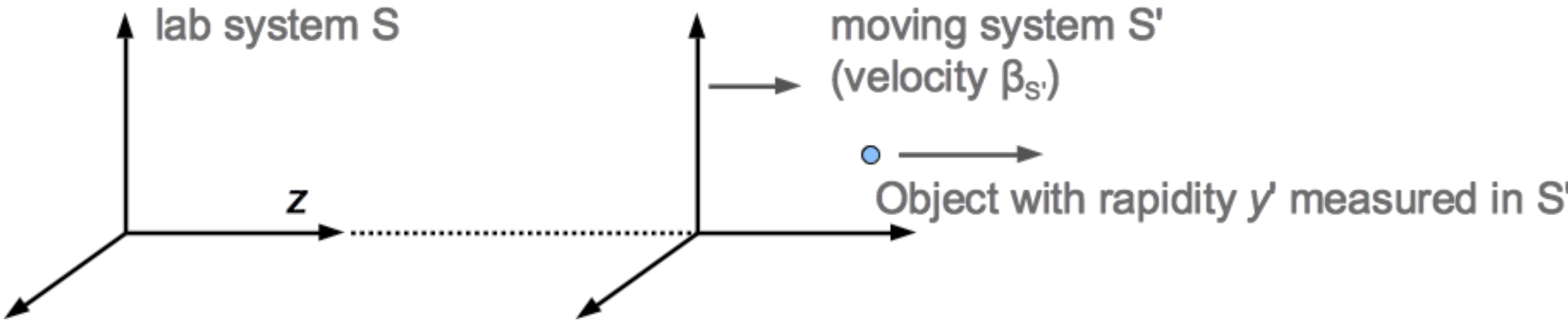
With $$e^y = \sqrt{\frac{E+p_L}{E-p_L}}, \qquad e^{-y} = \sqrt{\frac{E-p_L}{E+p_L}}$$

and $$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right), \quad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

one obtains $$E = m_T \cdot \cosh y, \qquad p_L = m_T \cdot \sinh y$$

where $m_T := \sqrt{m^2 + p_T^2}$ is called *transverse mass*

Additivity of Rapidity under Lorentz Transformation



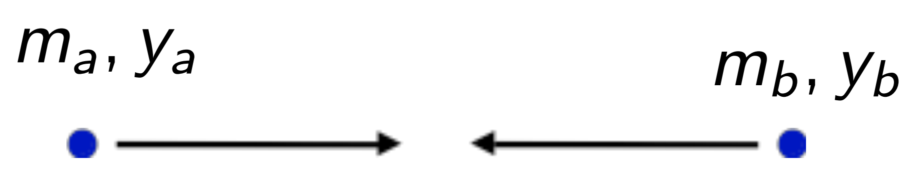
Lorentz transformation: $E = \gamma(E' + \beta p_z'), \qquad p_z = \gamma(p_z' + \beta E') \qquad (\beta \equiv \beta_{S'})$

$$
\begin{aligned}
y &= \frac{1}{2} \ln \frac{E + p_z}{E - p_z} \\
&= \frac{1}{2} \ln \frac{\gamma(E' + \beta p_z') + \gamma(p_z' + \beta E')}{\gamma(E' + \beta p_z') - \gamma(p_z' + \beta E')} \\
&= \frac{1}{2} \ln \frac{(1+\beta)(E' + p_z')}{(1-\beta)(E' - p_z')} \\
&= \underbrace{\frac{1}{2} \ln \frac{1+\beta}{1-\beta}}_{\text{rapidity of S' as measured in S}} + \underbrace{\frac{1}{2} \ln \frac{E' + p_z'}{E' - p_z'}}_{y'}
\end{aligned}
$$

y is not Lorentz invariant. however. it has a simple transformation property:

$$y = y' + y_{S'}$$

Rapidity of the CMS (I)



$$a = (E_a, 0, 0, p_a)$$
$$b = (E_b, 0, 0, -p_b)$$

Velocity of the CMS:

$$a_z^* = \gamma_{\mathrm{cm}}(a_z - \beta_{\mathrm{CM}} a_0) \stackrel{!}{=} -b_z^* = -\gamma_{\mathrm{cm}}(b_z - \beta_{\mathrm{CM}} b_0) \qquad \Rightarrow \beta_{\mathrm{cm}} = \frac{a_z + b_z}{a_0 + b_0}$$

Using the formula for the rapidity we obtain

$$y_{\mathrm{cm}} = \frac{1}{2} \ln \left[\frac{a_0 + a_z + b_0 + b_z}{a_0 - a_z + b_0 - b_z} \right]$$

Writing energies and momenta in terms of rapidity:

$$\begin{aligned} y_{\mathrm{cm}} &= \frac{1}{2} \ln \left[\frac{m_a e^{y_a} + m_b e^{y_b}}{m_a e^{-y_a} + m_b e^{-y_b}} \right] \\ &= \frac{1}{2}(y_a + y_b) + \frac{1}{2} \ln \left[\frac{m_a e^{y_a} + m_b e^{y_b}}{m_a e^{y_b} + m_b e^{y_a}} \right] \end{aligned}$$

Rapidity of the CMS (II)

For a collision of two particles with equal mass *m* and rapidities y_a and y_b. the rapidity of the CMS y_{cm} is then given by:

$$y_{\mathrm{cm}} = (y_a + y_b)/2$$

In the center-of-mass frame. the rapidities of particles a and b are:

$$y_a^* = y_a - y_{\mathrm{cm}} = -\frac{1}{2}(y_b - y_a) \qquad y_b^* = y_b - y_{\mathrm{cm}} = \frac{1}{2}(y_b - y_a)$$

Examples (CMS rapidity of the nucleon-nucleon system)

a) fixed target experiment: $y_{CM} = (y_{\mathrm{target}} + y_{\mathrm{beam}})/2 = y_{\mathrm{beam}}/2$

b) collider (same species and beam momentum): $y_{CM} = (y_{\mathrm{target}} + y_{\mathrm{beam}})/2 = 0$

c) collider (two different ions species. same *B* field):

$$y_{cm} = \frac{1}{2} \ln \frac{Z_1 A_2}{A_1 Z_2}$$

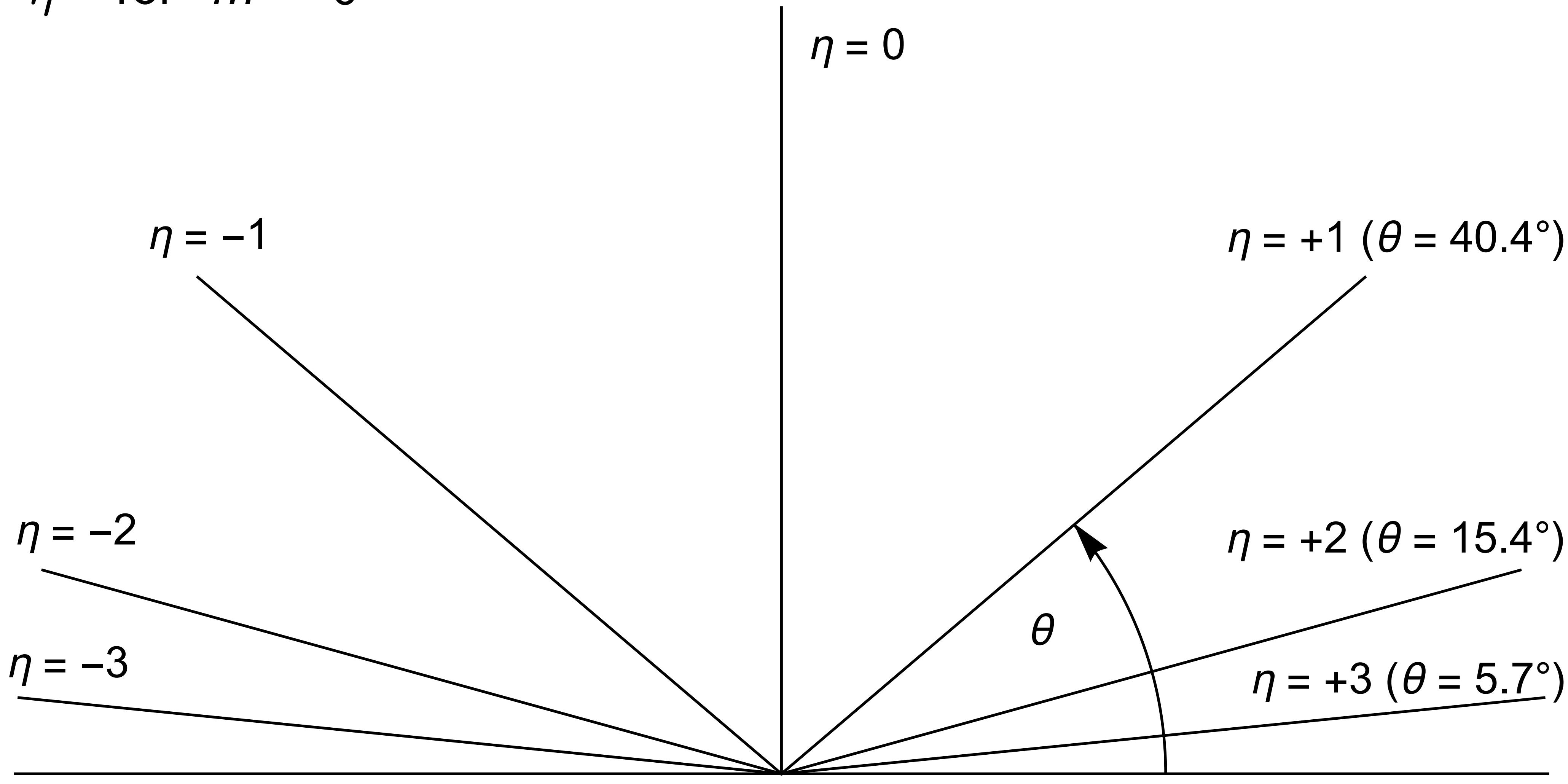
[exercise] p-Pb beam at LHC: $y_{CM} \approx 0.465$

Pseudorapidity η

$$\cos(2\alpha) = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$y = \frac{1}{2}\ln\frac{E + p\cos\vartheta}{E - p\cos\vartheta} \stackrel{p \gg m}{\approx} \frac{1}{2}\ln\frac{1+\cos\vartheta}{1-\cos\vartheta} = \frac{1}{2}\ln\frac{2\cos^2\frac{\vartheta}{2}}{2\sin^2\frac{\vartheta}{2}} = -\ln\left[\tan\frac{\vartheta}{2}\right] =: \eta$$

$y = \eta$ for $m = 0$



Analogous to the relations for the rapidity we find:

$$p = p_T \cdot \cosh\eta, \qquad p_L = p_T \cdot \sinh\eta$$

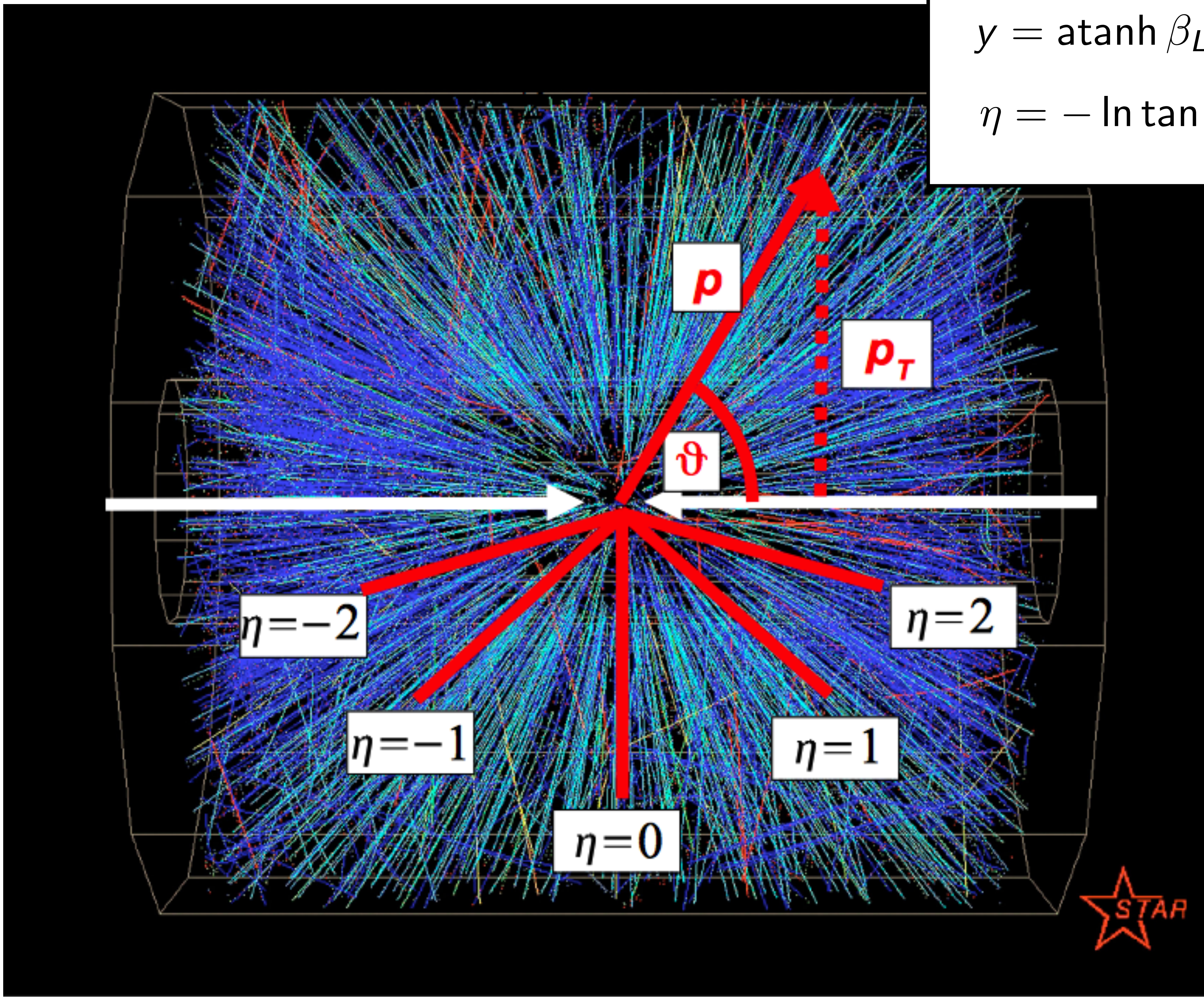
Example: Beam Rapidities

$$y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = \ln\frac{E+p_z}{\sqrt{E^2-p_z^2}} = \ln\frac{E+p_z}{m} \approx \ln\frac{2E}{m}$$

Beam momentum (GeV/*c*)	Beam rapidity
100	5.36
158	5.81
1380 (= 3500·82/208)	7.99
2760 (= 7000·82/208)	8.86
3500	8.92
6500	9.54
7000	9.61

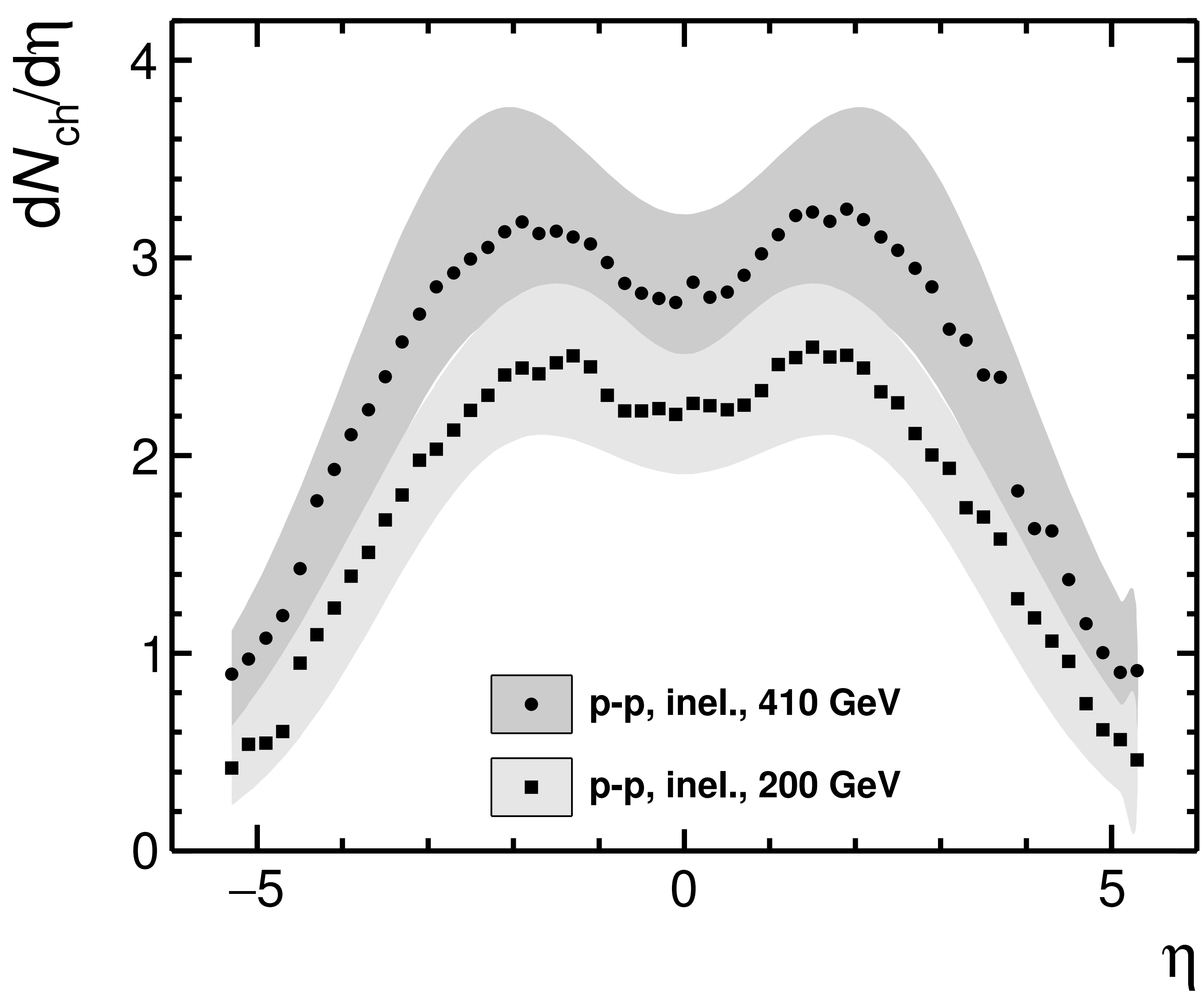
Brief summary

$$p_T = p \sin \vartheta$$
$$y = \operatorname{atanh} \beta_L$$
$$\eta = -\ln \tan \frac{\vartheta}{2}$$



Example of a Pseudorapidity Distribution of Charged Particles

PHOBOS. Phys.Rev. C83 (2011) 024913



Beam rapidity:

$$y_{\mathrm{beam}} = \ln \frac{E+p}{m} = 5.36$$

Average number of charged particles per collision (pp at $\sqrt{s}$ = 200 GeV):

$$\langle N_{ch} \rangle = \int \frac{dN_{ch}}{d\eta} d\eta \approx 20$$

Difference between dN/dy and dN/dη in the CMS

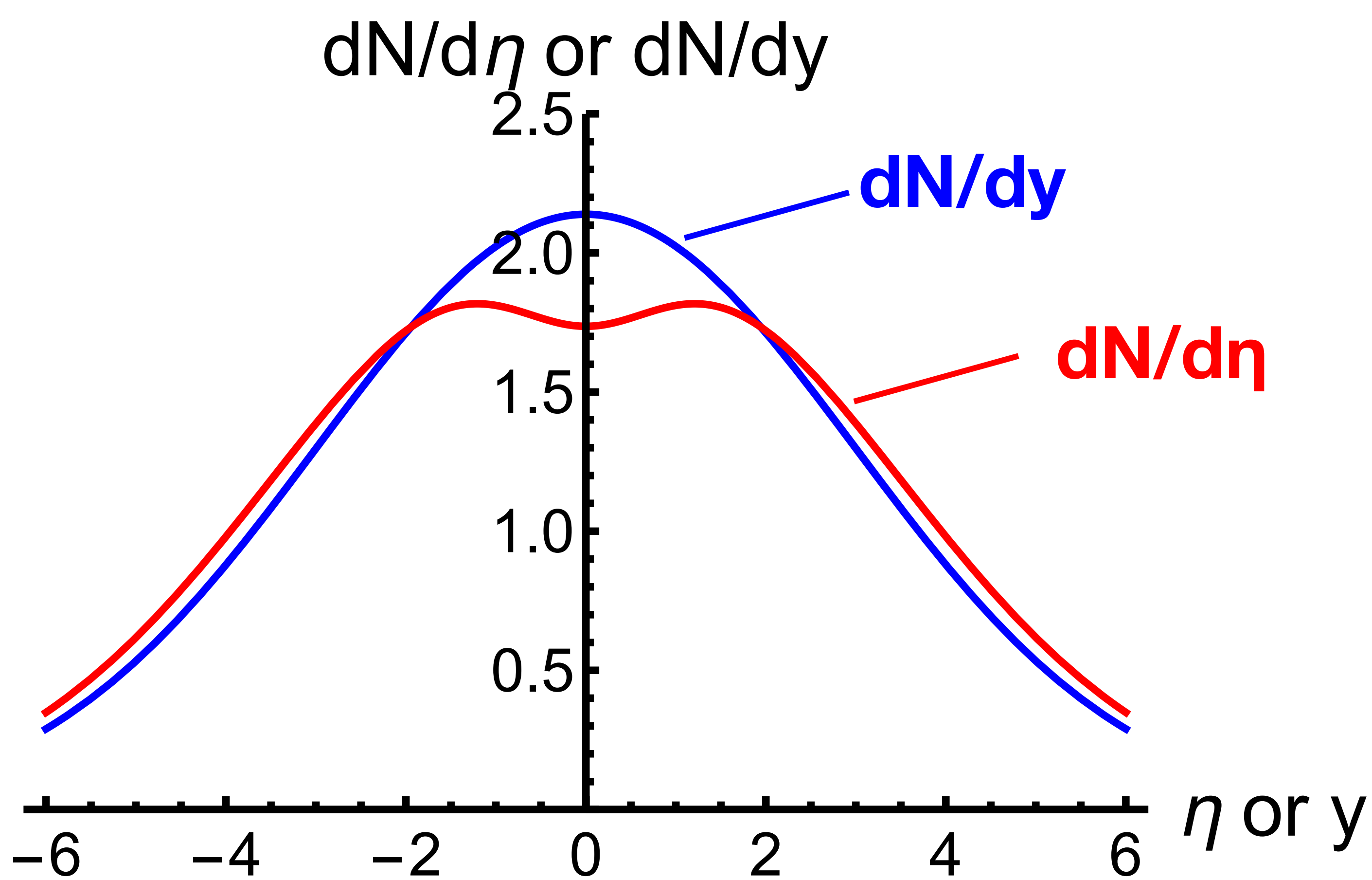
$$\frac{dN}{d\eta} = \sqrt{1 - \frac{m^2}{m_T^2 \cosh^2 y}}\,\frac{dN}{dy} \qquad y(\eta) = \frac{1}{2}\log\left(\frac{\sqrt{p_T^2\cosh^2\eta + m^2} + p_T\sinh\eta}{\sqrt{p_T^2\cosh^2\eta + m^2} - p_T\sinh\eta}\right)$$

Difference between dN/dy and dN/dη in the CMS at y = 0:

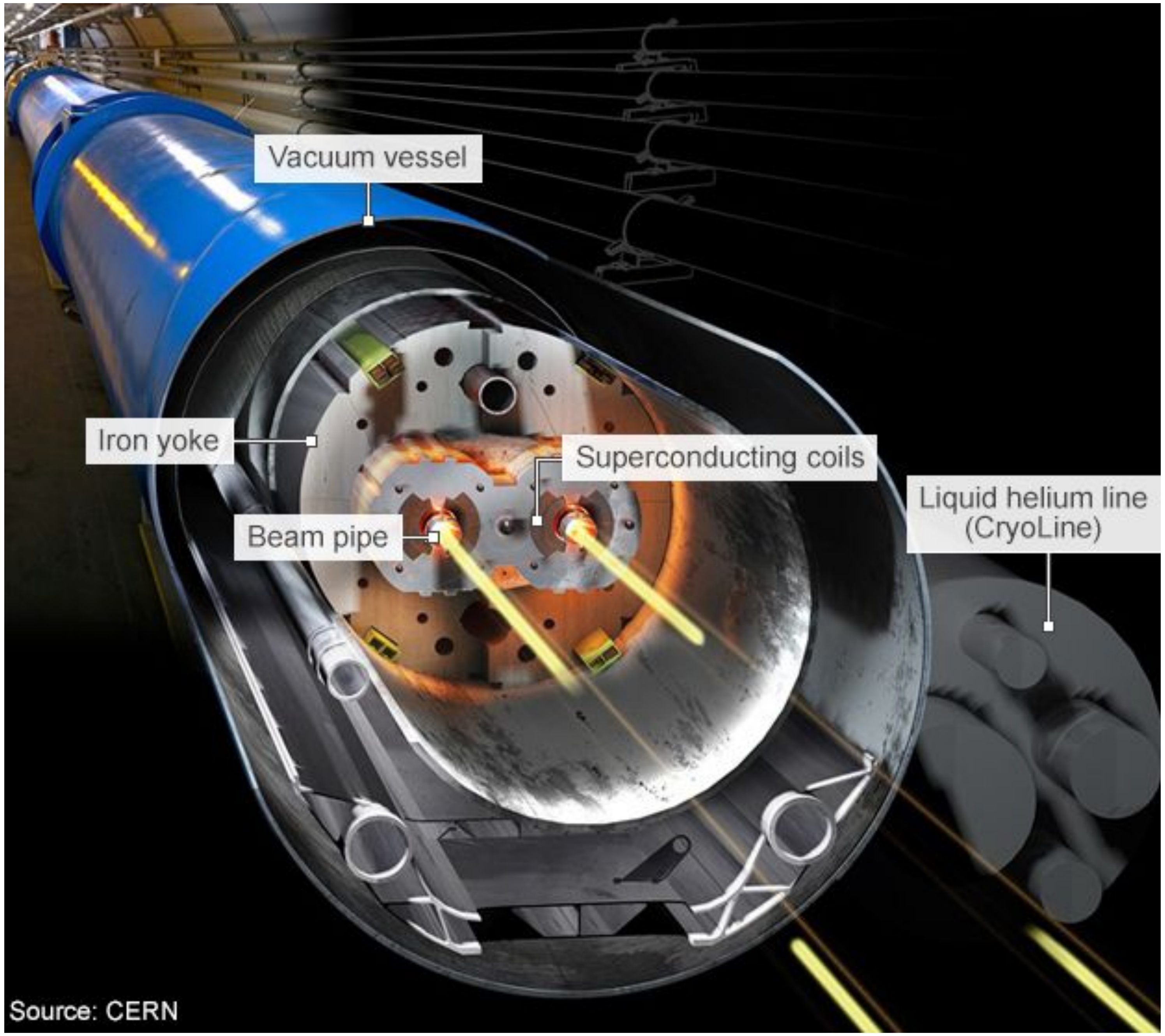
Simple example:
Pions distributed according to

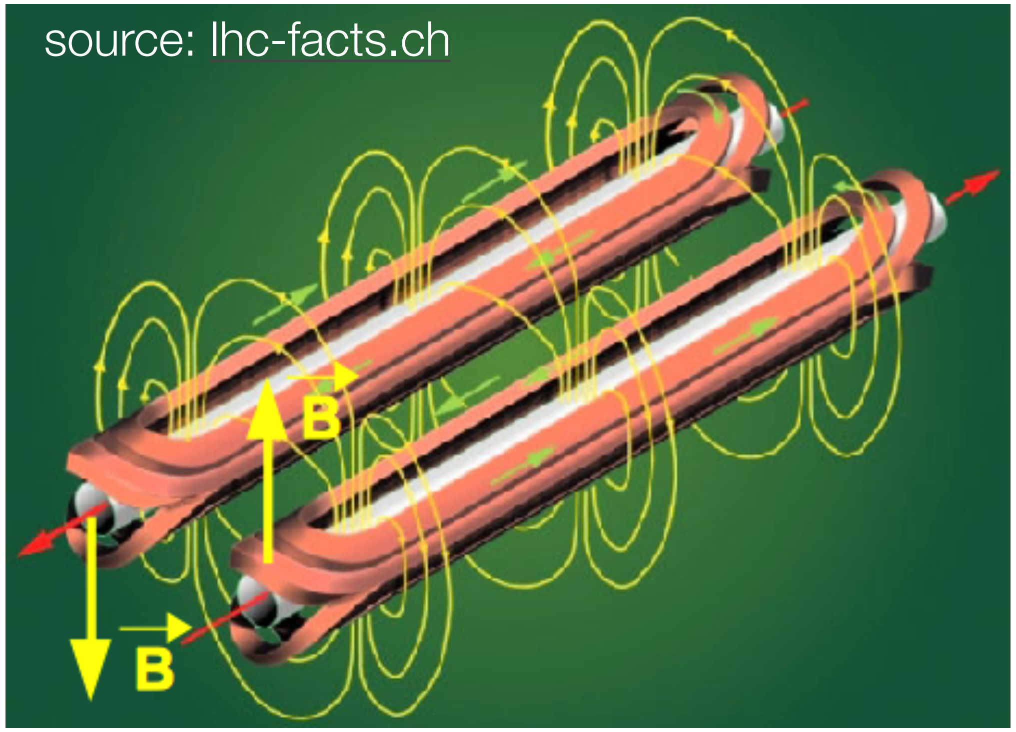
$$\frac{1}{2\pi p_T}\frac{d^2N}{dp_T dy} = G(y)\cdot\exp(-p_T/0.16)$$

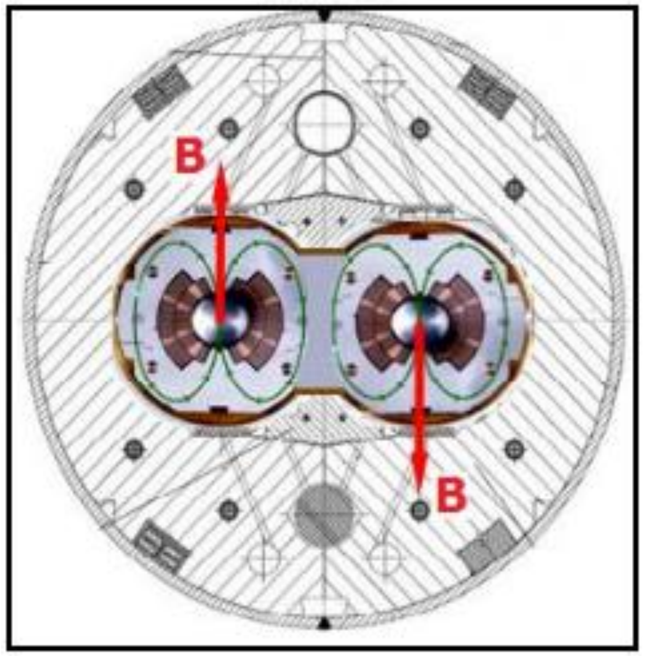
Gaussian with σ = 3 — p_T in GeV



LHC dipole


Vacuum vessel
Iron yoke
Superconducting coils
Liquid helium line
(CryoLine)
Beam pipe
Source: CERN


source: lhc-facts.ch
B
B


B
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LHC parameters

transverse beam radius: about 20 μm

	pp 2011	Pb-Pb 2011
Beam energy (per nucleon)	3.5 TeV	3.5 TeV·82/208
Particles/bunch	$1.35 \cdot 10^{11}$	$1.2 \cdot 10^{8}$
#bunches per beam	1380	358
Bunch spacing	50 ns (= 15 m)	200 ns
RMS bunch length	7.6 cm	9.8 cm
peak luminosity	$3.65 \cdot 10^{33}$ cm^{-2} s^{-1}	$0.5 \cdot 10^{27}$ cm^{-2} s^{-1}

https://home.cern/resources/brochure/accelerators/lhc-facts-and-figures
https://www.lhc-closer.es/taking_a_closer_look_at_lhc/1.lhc_parameters

Luminosity and cross section

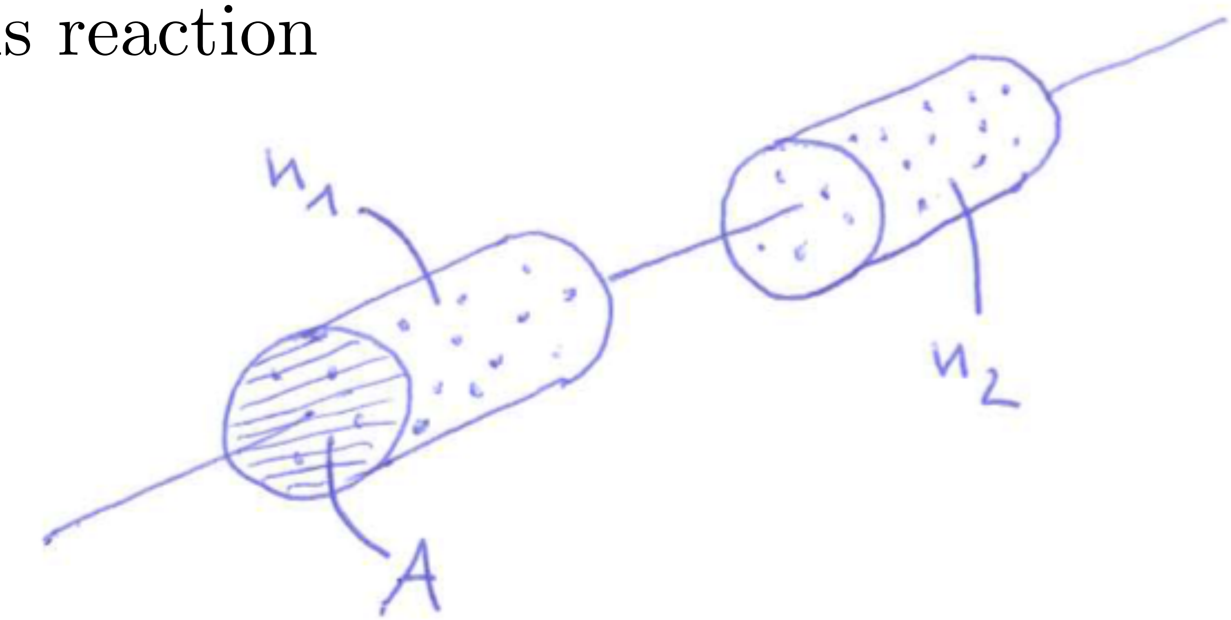
$$\frac{dN_{int}}{dt} = \sigma \cdot L$$

$$L = \text{luminosity (in s}^{-1}\text{cm}^{-2})$$

$$dN_{int}/dt = \text{Number of interactions of a certain type per second}$$

$$\sigma = \text{cross section for this reaction}$$

$$L = \frac{n_1 n_2 f_{coll}}{A}$$



$$n_1,\ n_2 = \text{numbers of particles per bunch in the two beams}$$

$$f_{coll} = \text{bunch collision frequency at a given crossing point}$$

$$A = \text{beam crossing area } (A \approx 4\pi\sigma_x\sigma_y)$$

Lorentz invariant Phase Space Element

Observable: Average density of produced particles in momentum space

$$\frac{1}{L_{\mathrm{int}}} \frac{d^3 N_A}{d^3 \vec{p}} = \frac{1}{L_{\mathrm{int}}} \frac{d^3 N_A}{dp_x dp_y dp_z}$$

However. the phase space density would then not be Lorentz invariant (see next slides for details):

$$\frac{d^3 N}{dp'_x dp'_y dp'_z} = \frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} \cdot \frac{d^3 N}{dp_x dp_y dp_z} = \frac{E}{E'} \cdot \frac{d^3 N}{dp_x dp_y dp_z}$$

Lorentz invariant phase space element: $\frac{d^3 \vec{p}}{E} = \frac{dp_x dp_y dp_z}{E}$

The corresponding observable is called Lorentz invariant cross section:

$$E \frac{d^3 \sigma}{d^3 \vec{p}} = \frac{1}{L_{\mathrm{int}}} E \frac{d^3 N}{d^3 \vec{p}} = \underbrace{\frac{1}{N_{\mathrm{evt,tot}}} E \frac{d^3 N}{d^3 \vec{p}}}_{\text{this is called the invariant yield}} \sigma_{\mathrm{tot}}$$

Lorentz invariant Phase Space Element: Proof of invariance

Lorentz boost along the z axis:

$$
\begin{aligned}
p'_x &= p_x \\
p'_y &= p_y \\
p'_z &= \gamma(p_z - \beta E), \qquad & p_z &= \gamma(p'_z + \beta E') \\
E' &= \gamma(E - \beta p_z), \qquad & E &= \gamma(E' + \beta p'_z)
\end{aligned}
$$

Jacobian: $$\frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} = \begin{vmatrix} \frac{\partial p_x}{\partial p'_x} & 0 & 0 \\ 0 & \frac{\partial p_y}{\partial p'_y} & 0 \\ 0 & 0 & \frac{\partial p_z}{\partial p'_z} \end{vmatrix}$$

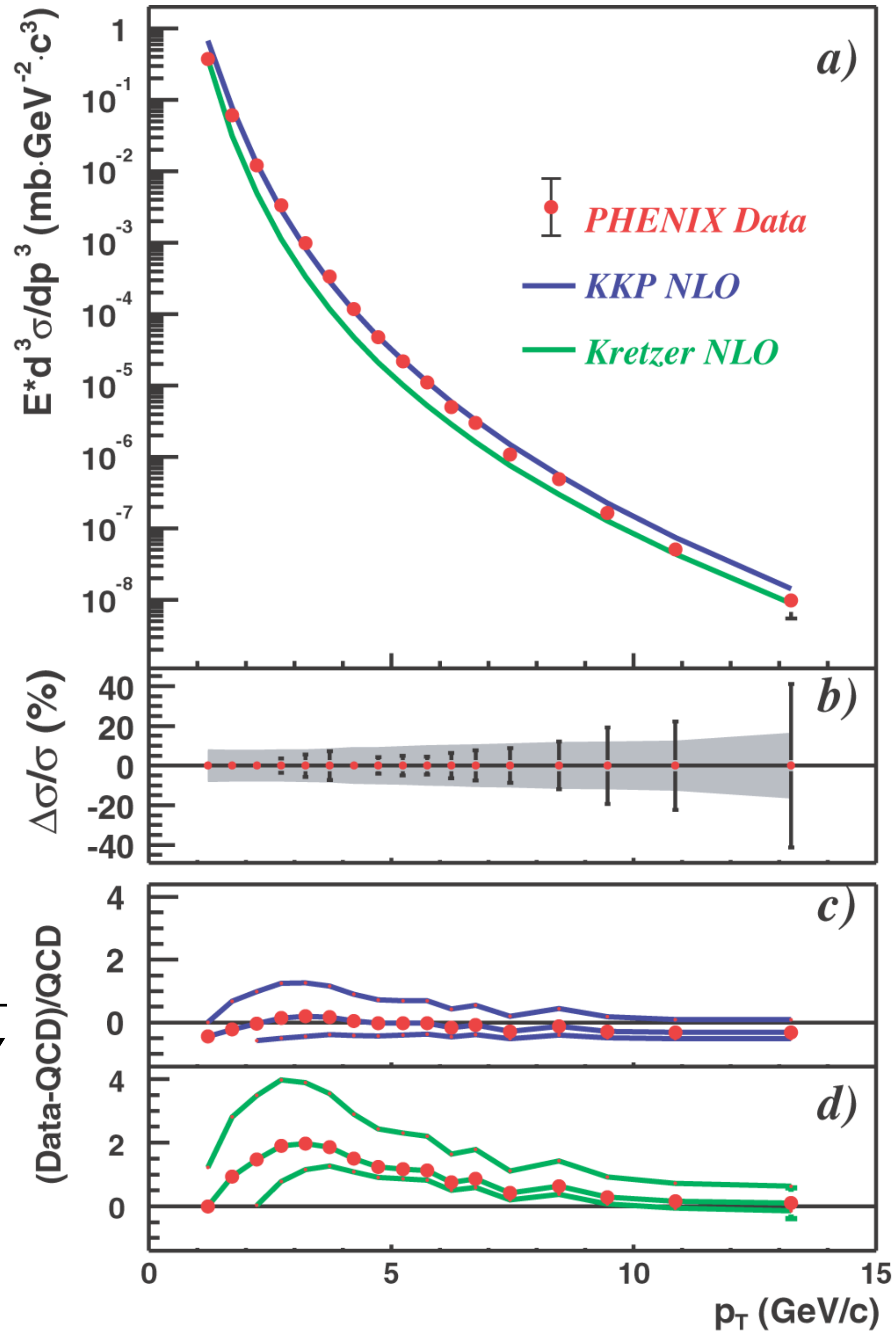
$$\frac{\partial p_x}{\partial p'_x} = 1, \quad \frac{\partial p_y}{\partial p'_y} = 1, \quad \frac{\partial p_z}{\partial p'_z} = \frac{\partial}{\partial p'_z}\left[\gamma\left(p'_z + \beta E'\right)\right] = \gamma\left(1 + \beta \frac{\partial E'}{\partial p'_z}\right)$$

$$\frac{\partial E'}{\partial p'_z} = \frac{\partial}{\partial p'_z}\left[\left(m^2 + p'^2_x + p'^2_y + p'^2_z\right)^{1/2}\right] = \frac{p'_z}{E'} \quad \rightsquigarrow \quad \frac{\partial p_z}{\partial p'_z} = \gamma\left(1 + \beta\frac{p'_z}{E'}\right) = \frac{E}{E'}$$

And so we finally obtain: $$\frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} = \frac{E}{E'}$$

Invariant Cross Section

Example: Invariant cross section for neutral pion production in p+p at $\sqrt{s}$ = 200 GeV



Calculation of the invariant cross section:

$$E\frac{d^3\sigma}{d^3p} = E\frac{1}{p_T}\frac{d^3\sigma}{dp_T\,dp_z\,d\varphi} \overset{dp_z/dy = m_T \cosh y = E}{=} \frac{1}{p_T}\frac{d^3\sigma}{dp_T\,dy\,d\varphi} \overset{\text{symmetry in } \varphi}{=} \frac{1}{2\pi p_T}\frac{d^2\sigma}{dp_T\,dy}$$

Sometimes also measured as a function of m_T:

$$\frac{1}{2\pi m_T}\frac{d^2\sigma}{dm_T\,dy} = \frac{1}{2\pi m_T}\frac{d^2\sigma}{dp_T\,dy}\frac{dp_T}{dm_T} = \frac{1}{2\pi p_T}\frac{d^2\sigma}{dp_T\,dy}$$

Integral of the inv. cross section

Average yield of particle X per event

$$\int E\frac{d^3\sigma}{d^3p}\,d^3p/E = \langle N_x\rangle \cdot \sigma_{\text{tot}}$$

Average path length of produced particles before decay

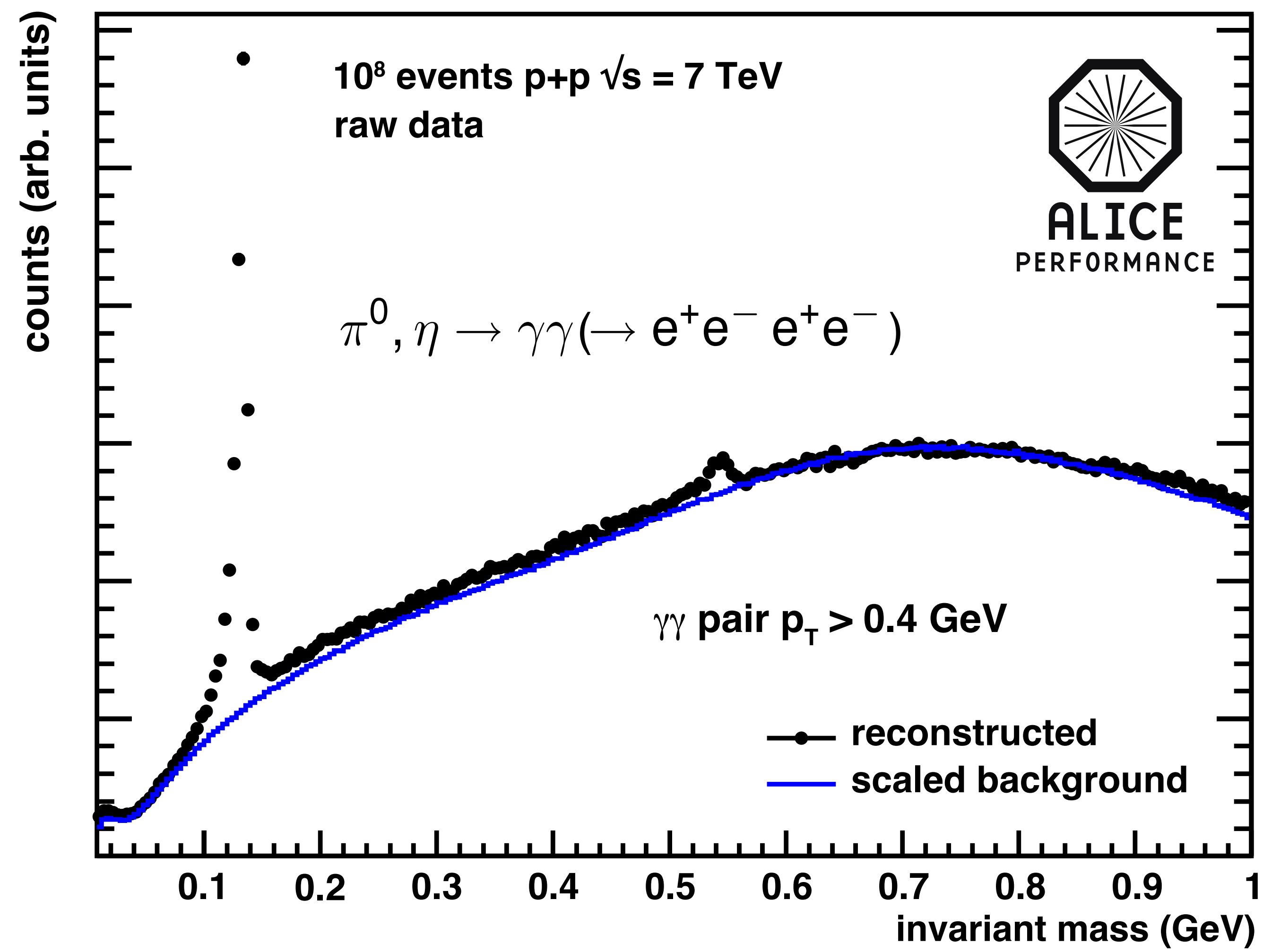
$$L_{\mathrm{lab}} = v \cdot \gamma \cdot \tau = \beta \cdot \gamma \cdot \tau \cdot c = \frac{p}{mc} \cdot \tau \cdot c$$

	mass (MeV)	mean life τ	$c\,\tau$	L_{lab} (p = 1 GeV/c)
π^+, π^-	139.6	$2.6 \cdot 10^{-8}$ s	7.80 m	56 m
π^0	135	$8.4 \cdot 10^{-17}$ s	25 nm	185 nm
K^+, K^-	494	$1.23 \cdot 10^{-8}$ s	3.70 m	7.49 m
K_S^0	497	$0.89 \cdot 10^{-10}$ s	2.67 cm	5.37 cm
K_L^0	497	$5.2 \cdot 10^{-8}$ s	15.50 m	31.19 m
D^+, D^-	1870	$1.04 \cdot 10^{-12}$ s	312 μm	167 μm
B^+, B^-	5279	$1.64 \cdot 10^{-12}$ s	491 μm	93 μm

Reconstruction of unstable particle via the invariant mass calculated from daughter particles

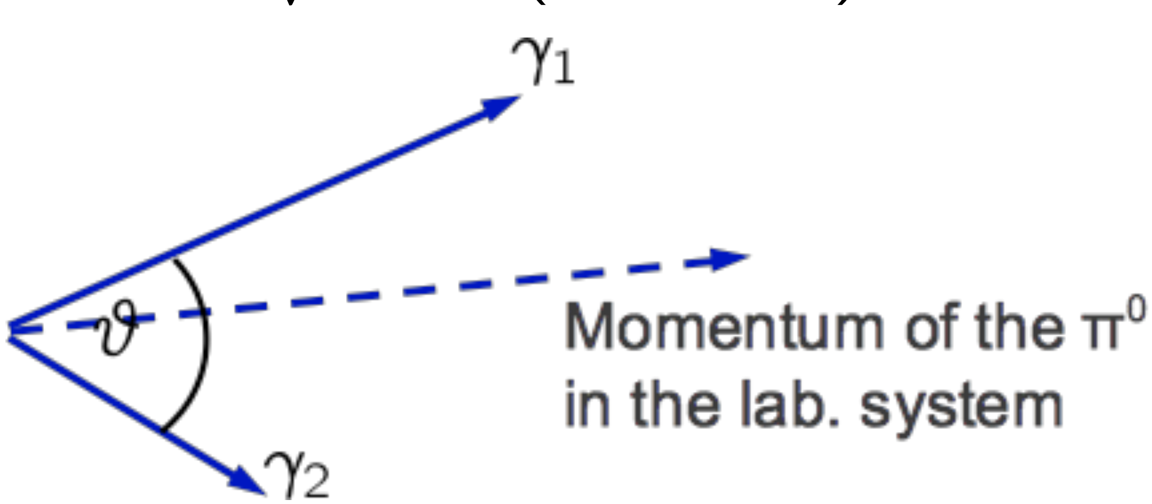
Consider the decay of a particle in two daughter particles. The mass of the mother particle is given by (“invariant mass”):

$$M^2 = \left[\begin{pmatrix} E_1 \\ \vec{p}_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ \vec{p}_2 \end{pmatrix} \right]^2$$
$$= m_1^2 + m_2^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2$$
$$= m_1^1 + m_2^2 + 2E_1E_2 - 2p_1p_2 \cos\vartheta$$



Example: π^0 decay:

$$\pi^0 \to \gamma + \gamma, \; m_1 = m_2 = 0, \; E_i = p_i$$
$$\Rightarrow M = \sqrt{2E_1E_2(1 - \cos\vartheta)}$$



Summary

- Center-of-mass energy $\sqrt{s}$:
  Total energy in the center-of-mass system (rest mass + kinetic energy)

- Observables: Transverse momentum p_T and rapidity y

- Pseudorapidity $\eta \approx y$ for $E \gg m$ ($\eta = y$ for $m = 0$. e.g.. for photons)

- Production rates of particles described by the Lorentz invariant cross section:

$$E \frac{d^3\sigma}{d^3p} = \frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T dy}$$