Exercises for the lecture „Moderne Methoden der Datenanalyse”

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Exercise 6: Lifetime fits

After performing unbinned maximum likelihood fits in the previous exercise sheet we will now look at χ^2-fits and binned likelihood fits. We will also perform multivariable fits and have a look at the correlations among these variables.

Exercise 6.1

Plot the exponentially distributed random numbers stored in the ntuple you created in Exercise 2.5. Use a histogram with 100 bins from 0 to 10.

Fit the following function to the distribution:

$$f_\tau(t) = N \cdot \exp\left(-\frac{t - t_0}{\tau}\right)$$

The fit parameters are the normalisation N, a possible offset t_0 and the lifetime τ. Use an object of type `TF1` to define the fit function. Fit the histogram using its `Fit` method with the default options (χ^2-fit). Try out different start values for the three parameters. How does the result of the fit depend on the start values? Plot the correlations between the fitted parameters. Consult the online documentation of `TH1::Fit` to find out how to obtain them. What can be learned from the correlations in this case?

Improve the parametrisation of the fit function. (Hint: Think about the p.d.f.) How does it have to look like in order to have the number of entries as parameter? Repeat the fit with the improved parametrisation.

Exercise 6.2

Make a histogram of the numbers from the ntuple with only 10 bins from 0 to 10. Fit the improved fit function to the histogram using once the default and once the option `"I"`. Compare the fitted parameters to the expected ones for both cases and explain the differences.

Exercise 6.3

Make three different histograms with 10, 1000 and all entries from the ntuple respectively. Use 1000 bins from 0 to 10. Fit the function to each histogram using the χ^2-method and the binned likelihood method. Have a look at the online documentation of the functions `FitLikelihood` and `FitLikelihoodI` in the source file of the class `TFitter` to find out how the likelihood is defined for a histogram. It is given by:

$$logL = \sum_{i=1}^{N}(n_i \cdot \ln \nu_i - \nu_i - \sum_{j=1}^{n_i} j)$$

where N is the number of bins, n_i is the entry in bin i and ν_i its expectation value.

When looking to the source code, note that large part of the code is to manipulate with histograms. It can also be useful to look to the function `TH1::Fit`. Compare the fitted parameters and the χ^2-value of both methods and explain the results.

Exercise 6.4*

The method of least-squares minimisation is commonly used to average N measurements of the same quantity. Table 1 lists four measurements, by different experiments, of the B-hadron[1] lifetime.

Measurement	Statistical error	Uncorrelated systematic error	Syst.1	Syst.2	Syst.3	Syst.4
1.511	0.022	0.053	0.042	0.006	0.011	0.037
1.542	0.021	0.034	0.016	0.007	0.008	0.023
1.611	0.010	0.024	0.008	0.000	0.010	0.000
1.564	0.030	0.031	0.018	0.005	0.004	0.009

Table 1: *Four independent measurements of the B hadron lifetime and their errors. All numbers are in units of ps.*

The measurement errors are split into *statistical* (i.e. random) errors and *systematic* errors which are uncertainties resulting from details of exactly how the measurements were made. These two types of error are independent of each other and so for each measurement, $\sigma^2_{tot} = \sigma^2_{stat} + \sigma^2_{syst}$. The systematic errors are further divided into an uncorrelated part plus contributions from four independent sources (labelled Syst. 1-4) which, because the measurement methods are similar, can be treated as being 100% correlated between the different measurements.

The average of these measurements, $\langle\tau\rangle$, can be estimated by minimising the following χ^2 function,

$$\chi^2(\langle\tau\rangle) = \sum_{i,j}^{N}(\tau_i - \langle\tau\rangle)V^{-1}_{ij}(\tau_j - \langle\tau\rangle) \quad (1)$$

where the τ_i are the measurement values and the matrix V is the covariance matrix between the values.

[1] A B-hadron consists of a B -type quark in a bound state with a lighter quark or quarks.

(a) Show that if the measurements are **uncorrelated**, minimising equation 1 and solving for $\langle \tau \rangle$ reduces to finding the simple weighted average of a set of measurements. Write a program to calculate the weighted mean and error.

(b) For the case where correlations between the measurement errors exist, the average is given analytically by, $\langle \tau \rangle = \sum_i^N w_i \cdot \tau_i$, where the weights are defined as

$$w_i = \frac{\sum_j^N \left(V^{-1}\right)_{ij}}{\sum_{k,l}^N \left(V^{-1}\right)_{kl}}$$

and the variance on the average is

$$V(\langle \tau \rangle) = \frac{1}{\sum_{k,l}^N \left(V^{-1}\right)_{kl}}$$

Write a program to calculate the correlated average and error. Implement the computation of the covariance matrix in the function `computeCov` which is predefined in the skeleton program.

Crosscheck the matrix inversion worked correctly by forming the matrix product $V \cdot V^{-1} = I$. You should also find that $\sum_i^N w_i = 1$. Print out the value of the weight assigned to each measurement and comment on what you find.

(c) Minimise the χ^2 of equation 1 using the Minuit package with $\langle \tau \rangle$ as the only free parameter. Follow the detailed instructions in the skeleton function `runMinuit`. Do you recover the same result as using the analytical solutions?