Exercises for the lecture „Moderne Methoden der Datenanalyse”

Prof. Dr. S. Hansmann-Menzemer, M. Schiller
Physikalisches Institut der Universität Heidelberg

June 22nd 2009

Exercise 7: Correlations

A common problem in science is the combination of several measurements. For the calculation of the average value not only the errors of the individual measurements have to be taken into account but also the correlations between them.

A wrong treatment of correlations or common systematic effects, however, can lead to biased results.

Exercise 7.1

At the LEP accelerator at CERN the mass of the W boson was measured in two different channels:

$$
\begin{aligned}
e^+e^- &\rightarrow W^+W^- \rightarrow q_1q_2q_3q_4 \\
e^+e^- &\rightarrow W^+W^- \rightarrow l\nu q_1q_2
\end{aligned}
$$

The experimental signature in the detector for the first channel with four quarks are four reconstructed jets. The second kind of reaction is identified by a lepton (electron or muon) and two jets. The neutrino is not detected. The measured W masses are:

$$
\begin{aligned}
\text{4 jets channel:} \quad m_W &= (80457 \pm 30 \pm 11 \pm 47 \pm 17 \pm 17)\ \text{MeV} \\
\text{lepton + 2 jets channel:} \quad m_W &= (80448 \pm 33 \pm 12 \pm \ 0 \pm 19 \pm 17)\ \text{MeV}
\end{aligned}
$$

The first two errors are the statistical and systematic experimental uncertainties. They are uncorrelated. The third error is an uncertainty from theory only present in the four jets channel. The fourth error is 100% correlated because it comes from a common theoretical model. Also the last error which originates from the LEP accelerator is 100% correlated between both measurements.

Construct a covariance matrix taking into account all uncertainties and their correlations. Use this covariance matrix to define a χ^2 expression containing the average W mass $\bar{m}_W$ as a free parameter. Determine $\bar{m}_W$ and its error by minimizing the χ^2 expression with the `TMinuit` class. This can be done in an easier way than we did it in previous exercises:

```
TMinuit* minuit = new TMinuit(numberOfParameters);
// Define parameters
// identifier, name, start value, step width, bounds
minuit->DefineParameter(0, "m", 80000, 50, 0, 0);
// tell Minuit which function to use
minuit->SetFCN(&FCN);
// run minimisation and error calculation
minuit->Migrad();
// get fitted parameters and error
minuit->GetParameter(0, par, sigma);
```

Exercise 7.2

Because the minimization of the χ^2 expression in exercise 7.1 is a linear problem it can be solved analytically. Determine $\bar{m}_W$ and its error analytically and compare them to the result of exercise 7.1.

Exercise 7.3

Estimate the contributions from statistical, systematic, theoretical and accelerator based uncertainties to the error of the combined W mass measurement. Use the quadratic difference between the total error and the error calculated with a covariance matrix where one component is removed.

Exercise 7.4

Two measurements $y_1 = 8.0$ and $y_2 = 8.5$ of the same physical quantity with an uncorrelated **relative** statistical error of 2 % and a common normalisation error of 10 % should be combined. Construct a covariance matrix and a χ^2 expression and determine its minimum with `TMinuit` or analytically.

Exercise 7.5

Is the result of exercise 7.4 reasonable? What could be the cause for the unexpected value? Make a plot of the covariance ellipse in the $y'_1 y'_2$ plane defined by

$$\Delta y^T V^{-1} \Delta y = c^2, \quad \Delta y = \begin{pmatrix} y_1 - y'_1 \\ y_2 - y'_2 \end{pmatrix}$$

for $c = 1$ and $c = 2$ together with the line $y'_1 = y'_2$. V is the covariance matrix. To draw the ellipse a `TGraph` object can be used. The points on the ellipse can be calculated as a function of the angle ϕ if Δy is expressed by ϕ and the radius r (see function `drawCovEllipse()`).

Exercise 7.6

Use an additional normalisation parameter N for the treatment of the common normalisation uncertainty in exercise 7.4 instead of taking it into account in the covariance matrix of y_1 and y_2. Add a term to the χ^2 expression for the normalisation with an expected value of 1 and an error of 10%. The normalisation factor N can be applied either to the measured values y_i or to the fit parameter

$\bar{y}$. Try out both ways (using `TMinuit` for the χ^2 minimisation) and compare the results. Which one is the correct result and why is the other one wrong?

Exercise 7.7

Determine $\bar{y}$ from the correct χ^2 expression of exercise 7.6 in an analytical way. How does the normalisation error affect the averaged value and its error?

Exercise 7.8

Construct a covariance matrix of y_1 and y_2 containing the normalisation uncertainty of 10% relative to the average value $\bar{y}$. Solve the corresponding χ^2 minimisation with `TMinuit` and plot the covariance ellipse.