Exercises for the lecture „Moderne Methoden der Datenanalyse"

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Exercise 9: Confidence Limits

"Is this a new discovery or just a statistical fluctuation?" Statistics offers some methods to give a quantitative answer. But these methods should not be used blindly. In particular one should know exactly what the obtained numbers mean and what they don't mean.

Exercise 9.1

Consider a counting experiment which can be modelled with a Poisson process with mean λ.

(a) Write a program that lists, for fixed values of $\lambda = 1, 2, \ldots$, the number of observed events n such that at most 10% (at least 90%) of the probability are above (below) n.

(b) Write a program that lists, for fixed number of observed events $n = 0, 1, 2, \ldots$, the upper and lower limits on λ at 90% confidence level. Why is there a difference to the numbers produced in (a)?

(Hint: To find the values of λ which solve the equations you get, you may want to use a scanning or a bisection approach.)

Exercise 9.2

Imagine you have a scales for weighting. Its resolution is $\sigma = 0.2$ g; assume the reading m follows a Gaussian distribution around the true weight μ.

(a) When measuring an apple, the reading is 175 g. Give an upper and a lower limit on the true mass, and also determine the central confidence interval (all at 90%). To calculate the integrals, use the identity:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_0^x e^{-\frac{x'^2}{2\sigma^2}} dx' = \frac{1}{2} \mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)$$

(b) When measuring a very light paper clip, the reading is -0.5 g. Give an upper and a lower limit on the true mass, and also determine the central confidence interval (at 90%). Does the result make sense?

(c) Try applying Bayesian statistics to the problem posed in (b). Applying Bayes' theorem, one has:

$$P(\mu|m) = \frac{P(m|\mu)}{P(m)} P(\mu)$$

Putting the Gaussian in, this becomes:

$$P(\mu|m) = \frac{e^{-\frac{(m-\mu)^2}{2\sigma^2}}}{\int_0^\infty e^{-\frac{(m-\mu')^2}{2\sigma^2}} d\mu'} \theta(\mu)$$

Use this PDF for μ to obtain upper and lower limits and a central interval at 90%.

Exercise 9.3

In a counting experiment 5 events are observed while $\lambda_B = 1.8$ background events are expected. Is this a significant (=3σ) excess? Calculate the probability of observing 5 or more events when the expectation value is 1.8 using Poisson statistics.

The number of signal events n_S can be treated as a Poisson variable with mean λ_S. The number of background events n_B can also be treated as a Poisson variable with mean λ_B, which we will assume for the moment to be known without error. The total number of events $n = n_S + n_B$ is therefore a Poisson variable with mean $\lambda = \lambda_S + \lambda_B$. The probability to observe n events is then:

$$p(n; \lambda_S; \lambda_B) = \frac{(\lambda_S + \lambda_B)^n}{n!} \exp(-[\lambda_S + \lambda_B])$$

We can now compute how likely it is to find n_{obs} or more events from the background alone, i.e. that a observed excess is due to a statistical fluctuation from the background. This is given by:

$$p(n \geq n_{obs}) = \sum_{n=n_{obs}}^{\infty} p(n; \lambda_S = 0; \lambda_B) = 1 - \sum_{n=0}^{n_{obs}-1} \frac{\lambda_B^n}{n!} \exp(-\lambda_B)$$

Exercise 9.4

Determine an upper limit λ_S^{max} for the number of signal events at a 95 % confidence level for the experiment from exercise 9.3. Such a limit is defined by the expected number of signal events λ_S^{max} where the probability of measuring the observed number of events or less reaches 5 % assuming a Poisson with mean $\lambda_B + \lambda_S^{max}$. Perform an interval search starting from the probabilities to observe $n_B + n_S^{min}$ and $n_B + n_S^{max}$ respectively or less events. Stop the search when the difference of the limits of the interval is less than 0.00001.

Exercise 9.5

Verify the limit determined in exercise 9.4 with toy experiments. In each toy experiment generate a random number according to a Poisson distribution with a mean value of $\lambda_B + \lambda_S^{max}$. Then count the number of experiments in which this random number is less or equal n_{obs}. The fraction of these events should be 5 %.