High Energy Frontier – Recent Results from the LHC: Heavy Ions II

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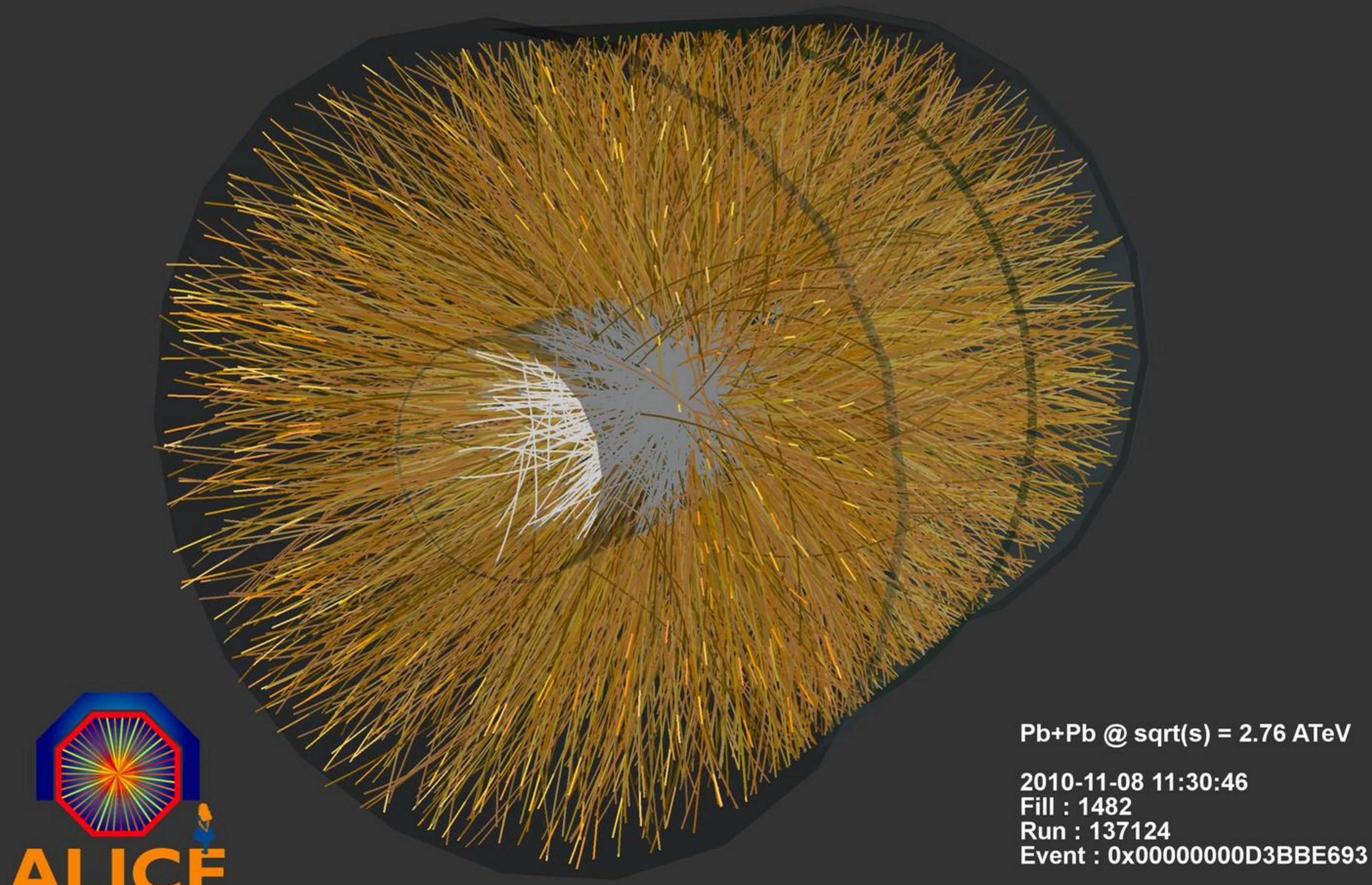
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Outline

- lecture 1 (22.11.): introduction
  - basics of relativistic heavy-ion collisions
- lecture 2 (29.11.): soft probes
  - hadron yields & spectra
  - hydrodynamics & collective motion
- lecture 3 (13.12.): hard probes
  - jets
  - heavy-flavor hadrons
- lecture 4 (20.12.): quarkonia & el.magn. probes
  - quest for J/ψ suppression/enhancement
  - direct & thermal photons
  - dileptons

Soft versus hard (pp)

- systematics of p_T spectra of charged particles versus $\sqrt{s}$
  - low p_T (below 2 GeV/c)

$$\frac{1}{p_T}\frac{dN}{dp_T} = A\left(\sqrt{s}\right)\cdot e^{-6\,p_T}$$

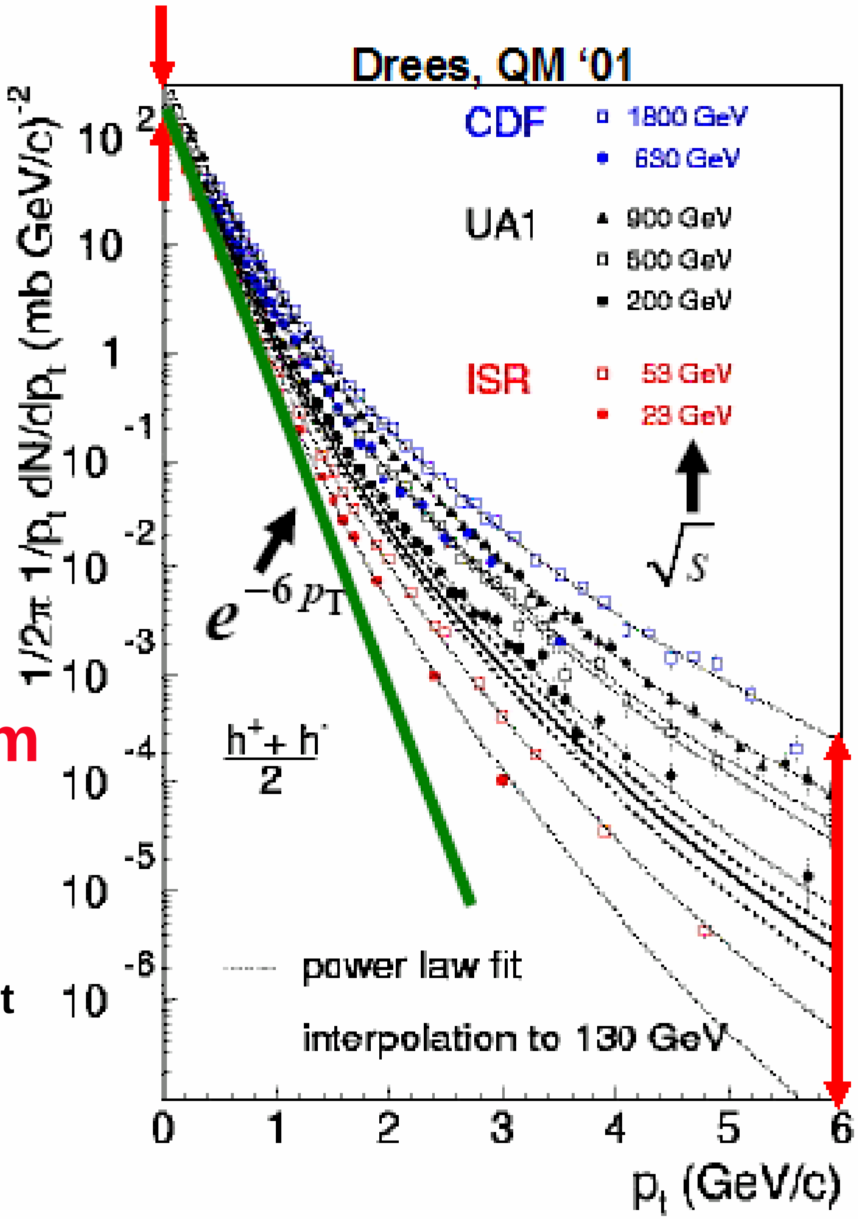
  - high p_T

$$\frac{1}{p_T}\frac{dN}{dp_T} = A\left(\sqrt{s}\right)\cdot \frac{1}{p_T^{\,n}}$$

  - mean transverse momentum

$$\langle p_T \rangle = \frac{\int_0^\infty p_T \frac{dN}{dp_T}\,dp_T}{\int_0^\infty \frac{dN}{dp_T}\,dp_T} \approx 300 - 400 \ \text{MeV}$$

(almost independent on $\sqrt{s}$)



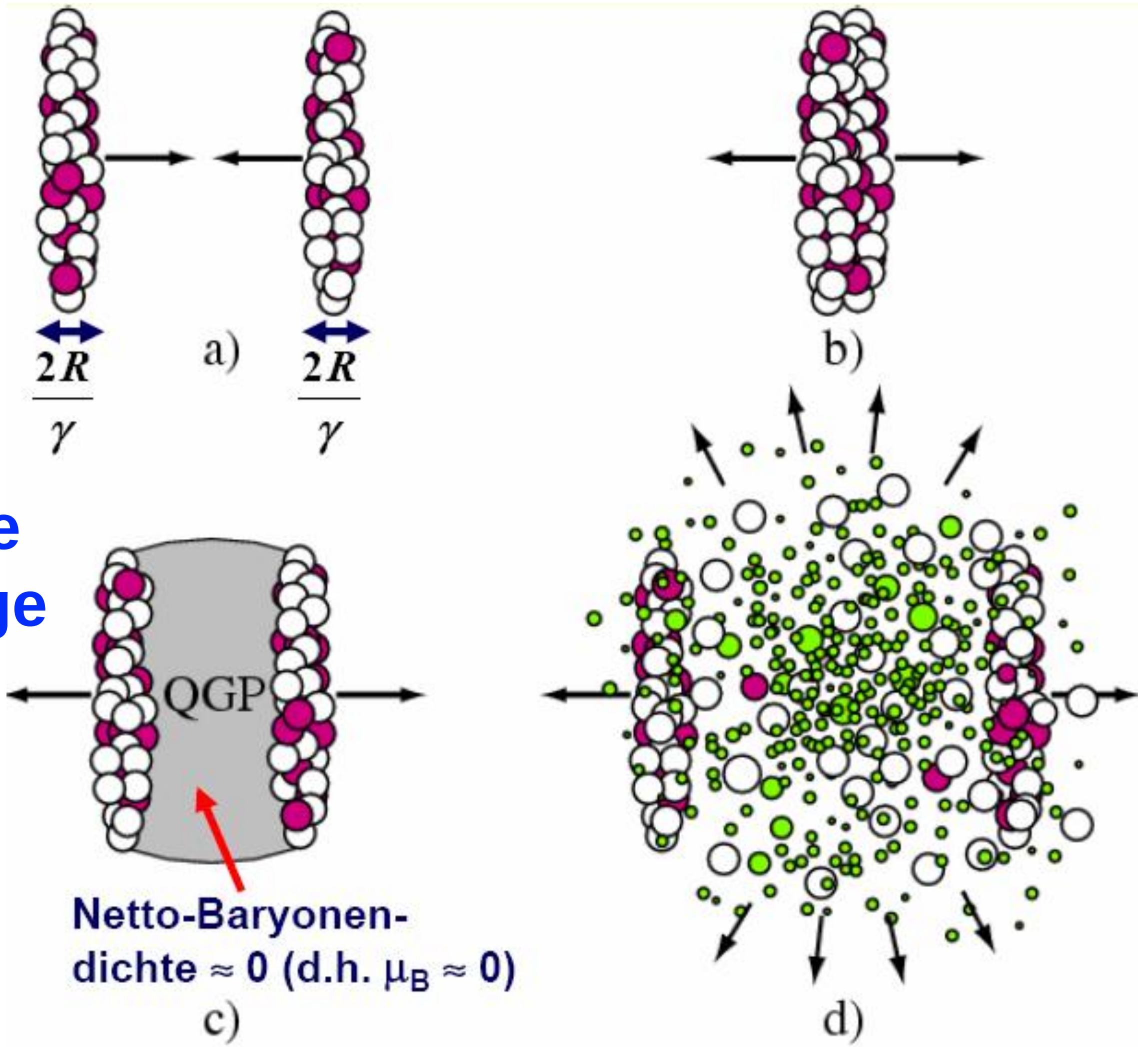
Nucleus-Nucleus collisions

a) Lorentz-contracted nuclei approach each other (γ_{RHIC}≈106, γ_{LHC}≈5860)

b) nuclei lose energy and penetrate each other

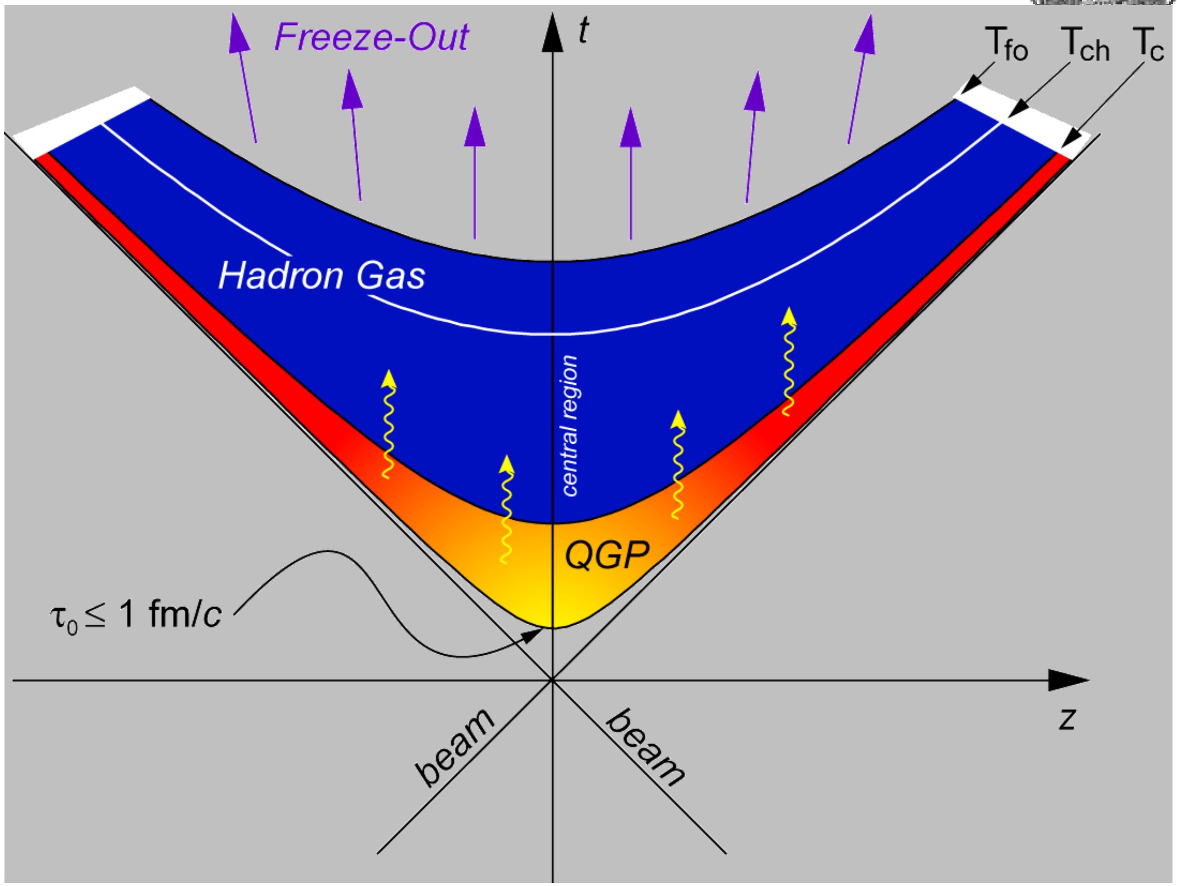
c) nuclear remnants leave collision zone with large energy density (QGP?)

d) energy density manifests itself in particle production



Freeze-Out in statistical models

- chemical freeze-out
  - inelastic interaction stop
  - no further change of different particle species yields



- thermal (kinetic) freeze-out
  - elastic interactions stop
  - no further change of kinematic distributions

Basics of statistical model

- assume a system in thermal (T = const.) and chemical (n_i = const.) equilibrium
- hadron gas: grand kanonical ensemble (system can exchange heat and particles with the environment, quantum numbers are conserved only on average)

$$n_i = \frac{g_i}{2\pi^2} \cdot \int_0^\infty \frac{p^2 dp}{\exp\left[\left(E_i - \mu_i\right)/T \pm 1\right]}, \qquad \mu_i = \mu_B B_i + \mu_S S_i + \mu_{I_3} I_{3,i}$$

- conserved quantum number → chem. potential μ
  - baryon number: $V \sum_i n_i B_i = Z + N \quad \rightarrow \quad V$
  - strangeness: $V \sum_i n_i S_i = 0 \quad \rightarrow \quad \mu_S$
  - charge: $V \sum_i n_i I_{3,i} = \frac{Z - N}{2} \quad \rightarrow \quad \mu_{I_3}$

2 free para-meters μ_B, T

Determination of (μ_B, T)

- particle densities in hadro-chemical equilibrium

$$N_h = g_h V \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\left(\sqrt{p^2+m_h^2}-\mu_B\right)/T} \pm 1}$$

$h = \pi, \eta, K, K^*, p, d, \Lambda, \Delta, \Xi, \Omega, D, \ldots\ldots$ and antiparticles

spin-isospin degeneracy

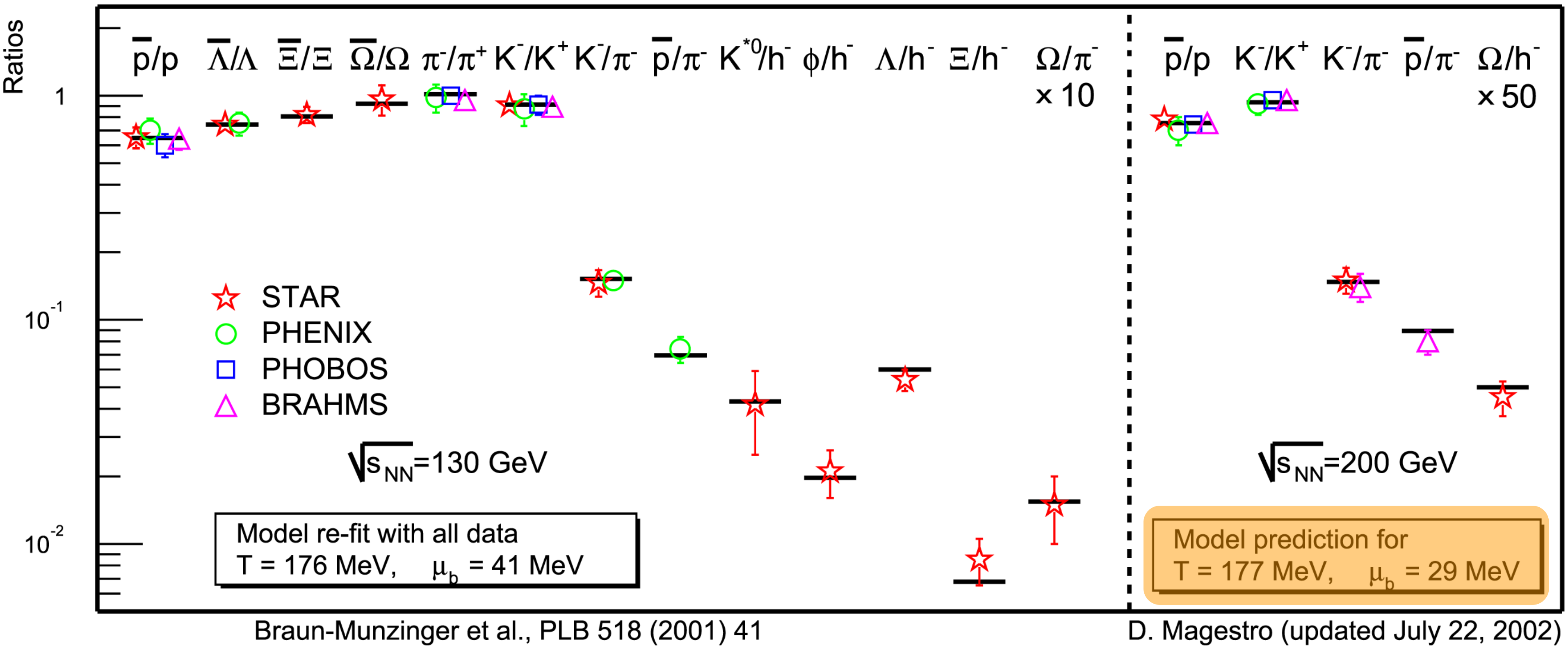
baryochem. potential

temperature

- one particle yield ratio (e.g. $\bar{p}/p$) determines μ_B/T
- a second ratio (e.g. π/p) determines T
- prediction of ALL other ratios

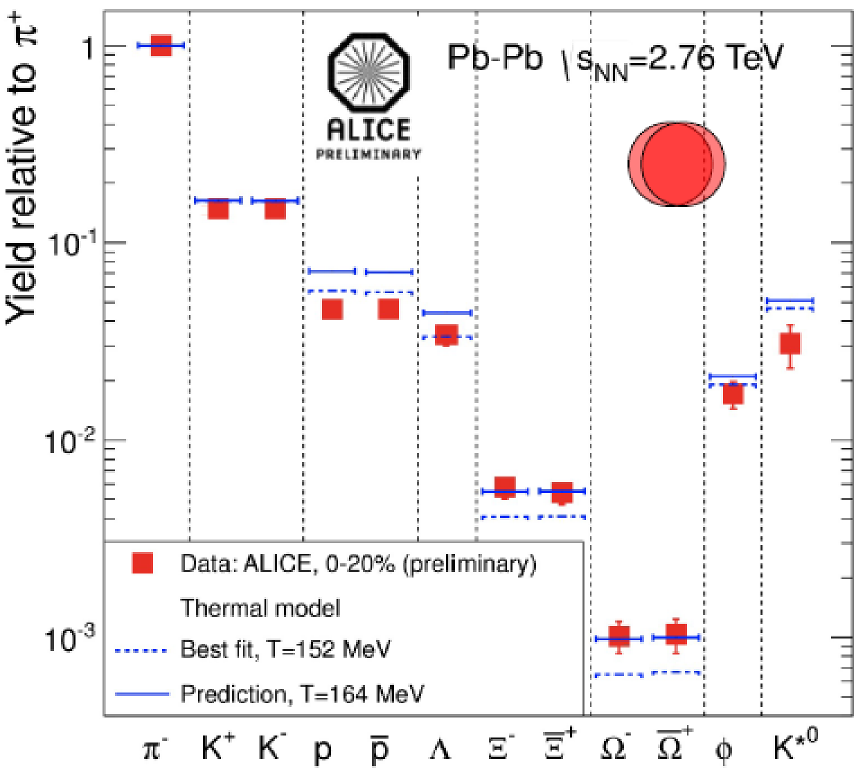
Determination of (μ_B, T)

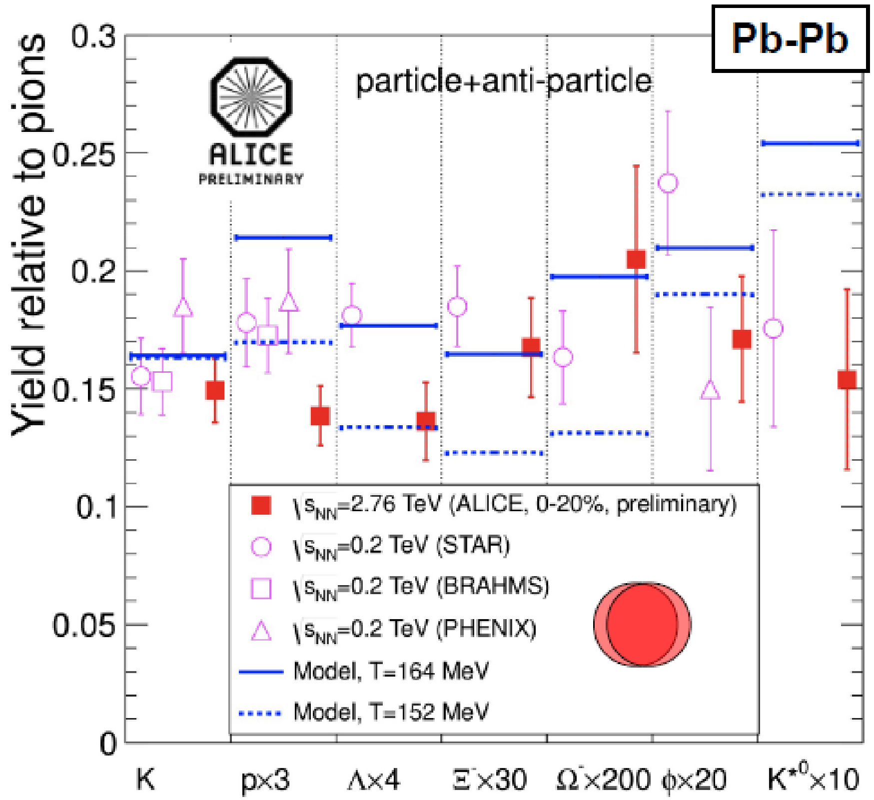
- particle densities in hadro-chemical equilibrium



Hadron yields at the LHC

- **works reasonably well**
  - **proton and lambda yields are ‚low‘ (still puzzling!)**

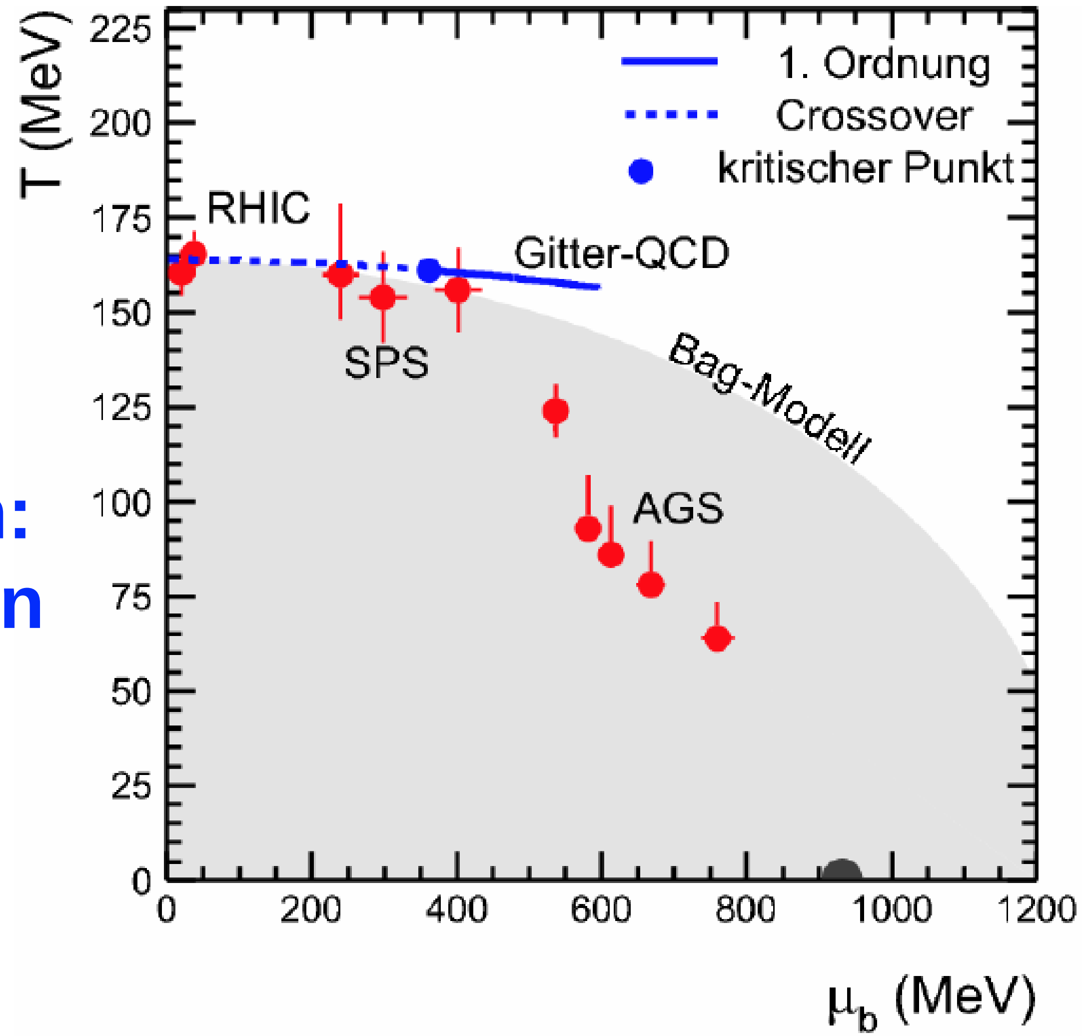




- **statistical model: A. Andronic et al., NPA 772(2006)167**

Energy dependence of (μ_B, T)

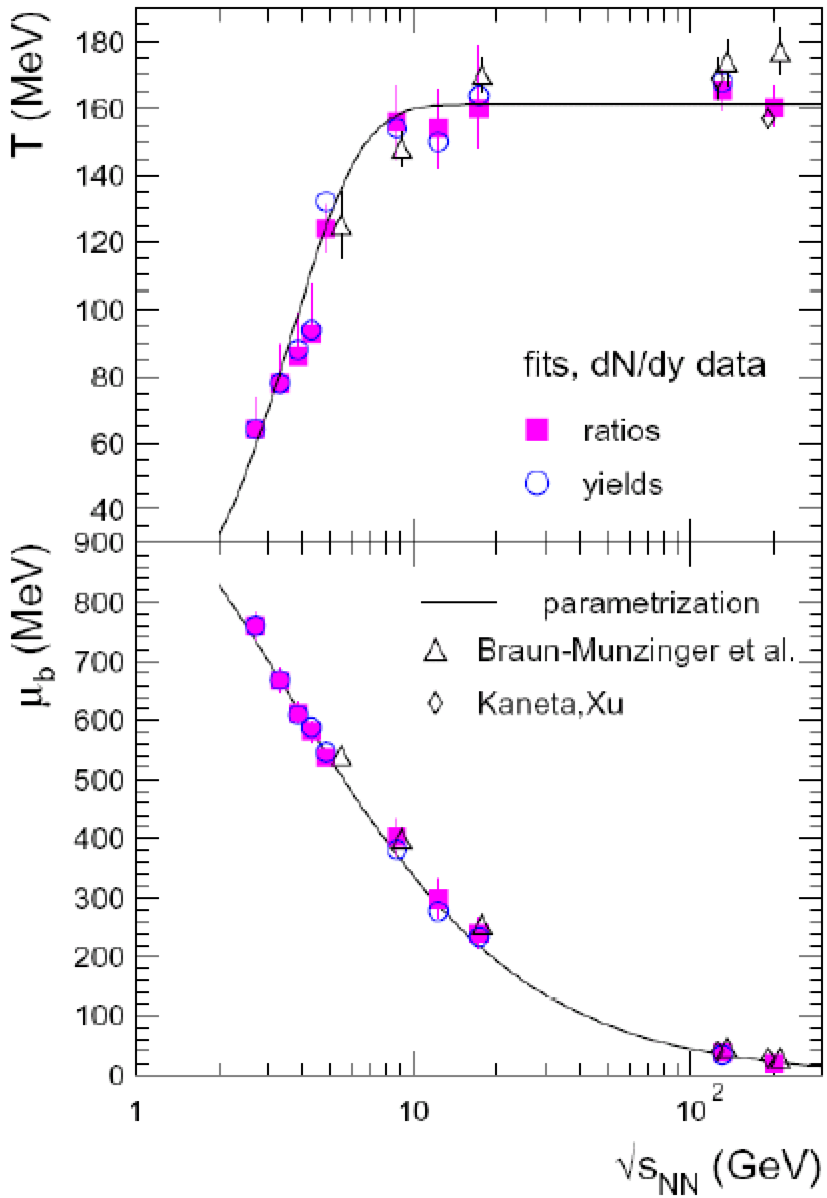
- (μ_B, T) from hadro-chemical model fits on phase boundary for SPS energies and above
- potential explanation: chemical composition is determined at the phase boundary
- AGS energies: investigation of compressed baryonic matter → FAIR @ GSI



Energy dependence of (μ_B, T)

- temperature
  - saturation above ~8 GeV
- baryo chemical potential
  - approaches zero towards higher energy

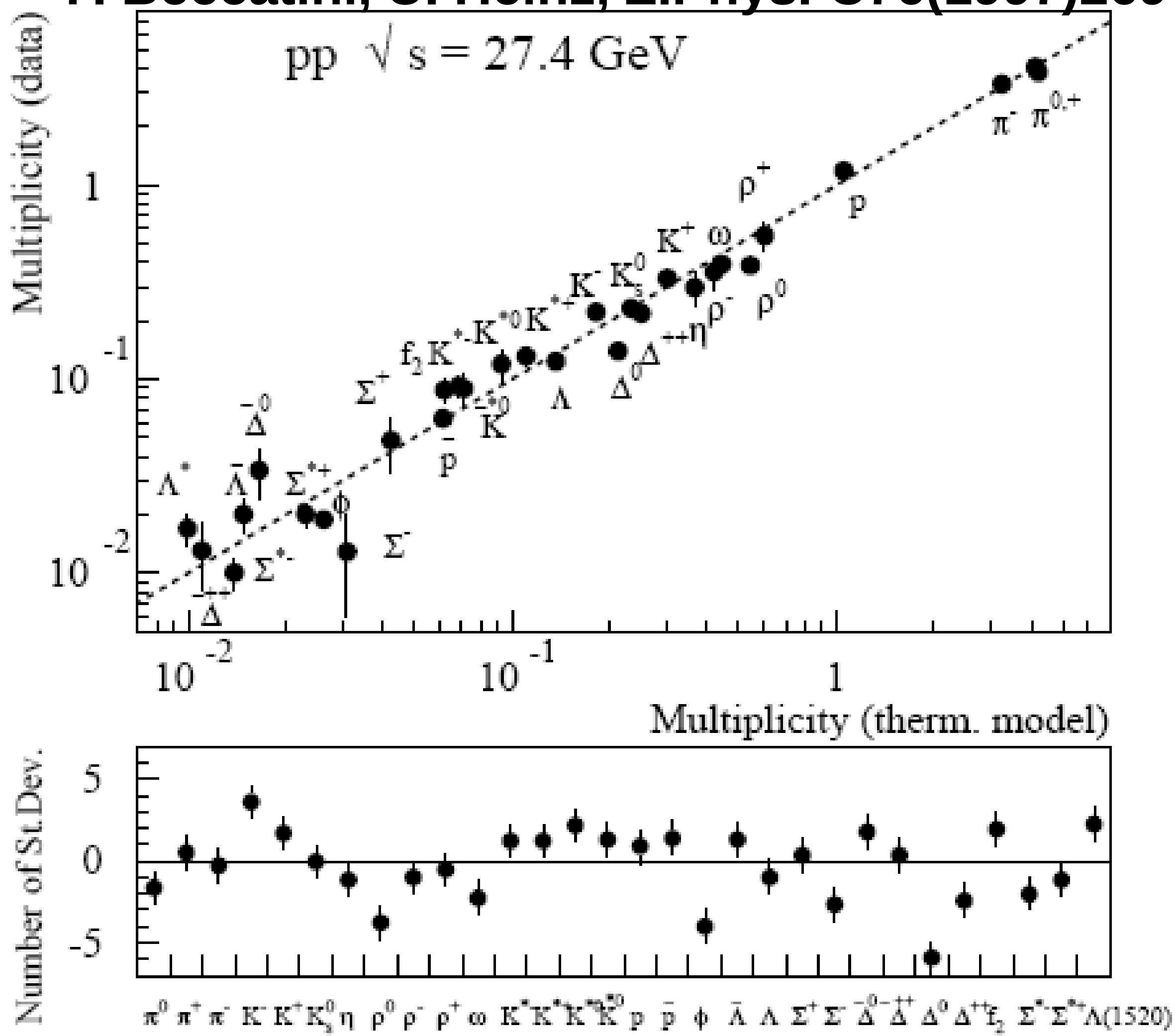
→ conditions at RHIC and the LHC approach those of the early universe



Thermalization in pp collisions

- **hadro chemical fit: not too bad?!**

F. Beccatini, U. Heinz, Z.Phys. C76(1997)269

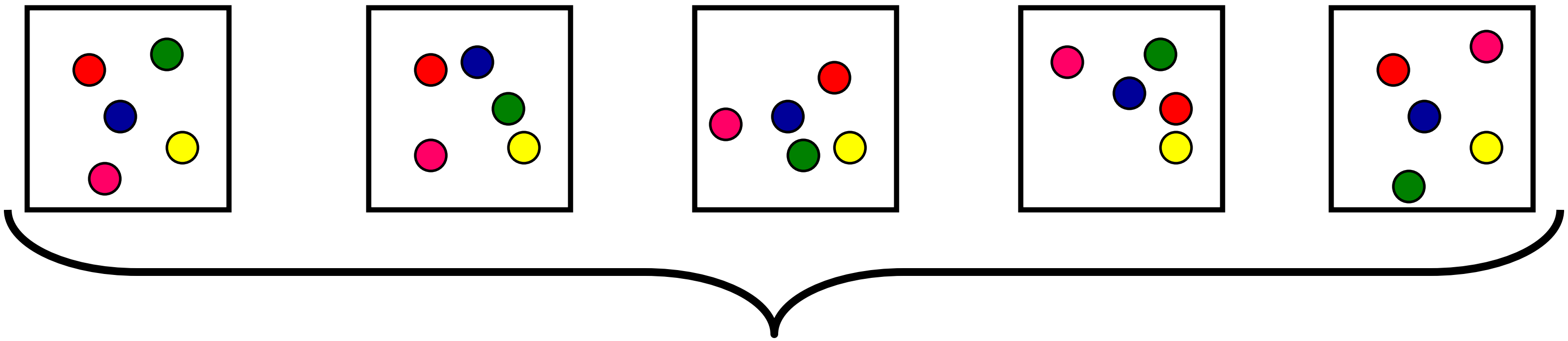


Thermalization in pp collisions

- **every process producing hadrons, which populates phase space uniformly, can not be distinguished from particle production from a micro canonical ensemble!**
  - **particle yield ratios follow from the populated phase space volumes**
  - **interactions between particles are not needed to populate phase space uniformly**
  - **thermal and/or chemical equilibrium are not necessary**

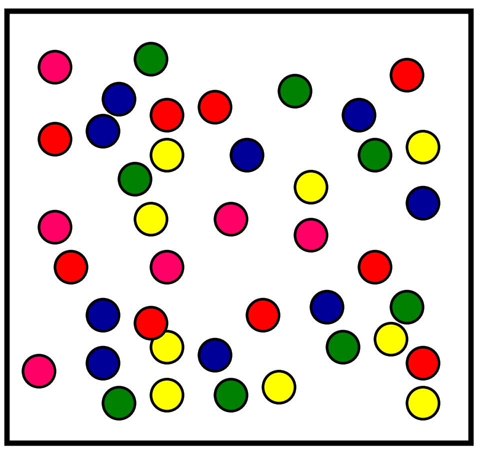
Statistics ≠ Thermodynamics

- pp collisions



ensemble of collisions = statistical ensemble

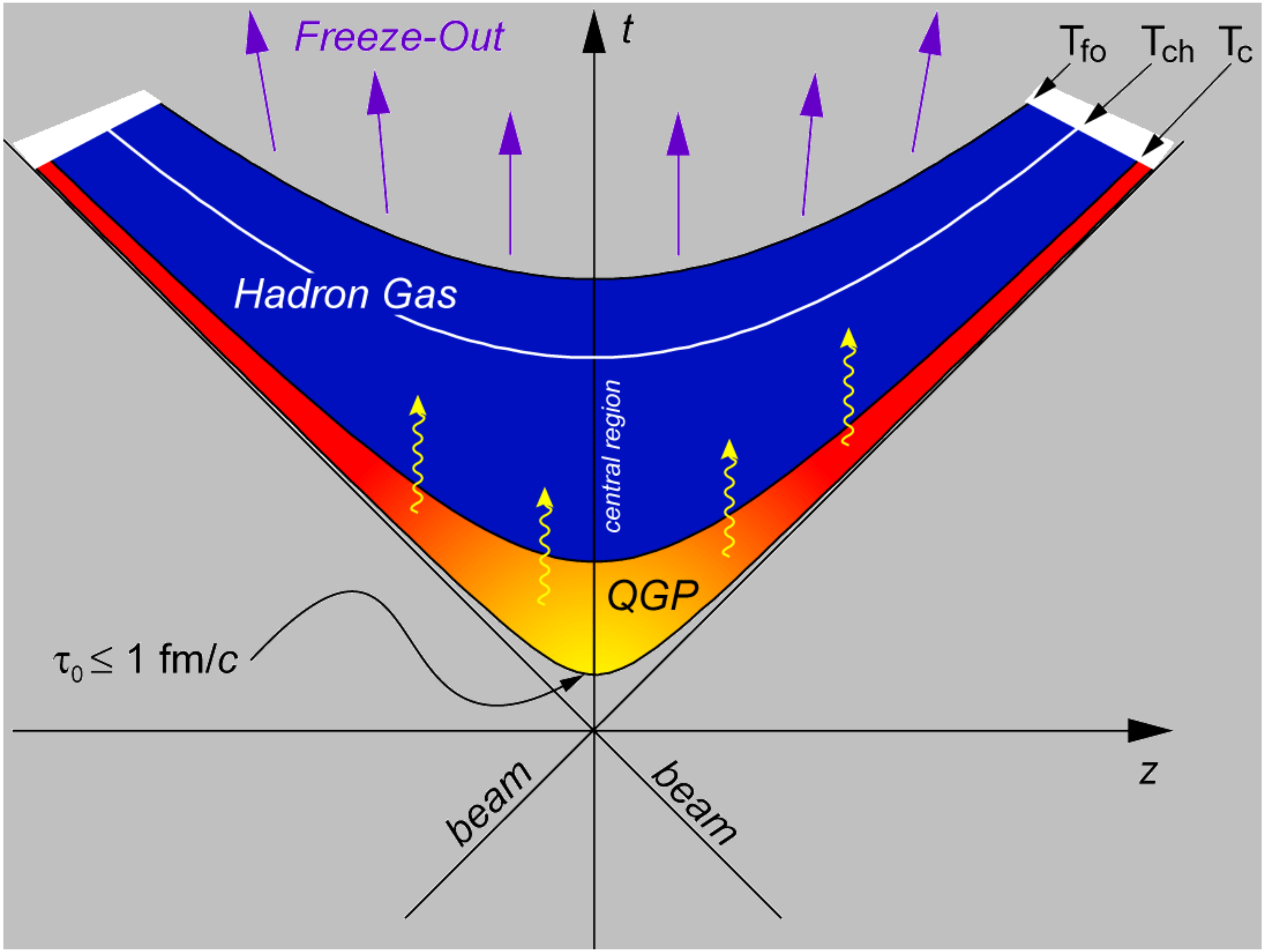
- AA collisions



- interaction of particles within collisions
- thermalization becomes possible
- pressure can become a meaningful concept

→ search for collective behaviour

Evolution of an AA collision



- how does the fireball develop in time?

Classical hydrodynamics

- describe evolution of a fluid
  - time dependent velocity profile: $\vec{u}(\vec{r},t)$
  - stationary case: $\vec{u}(\vec{r},t) \equiv \vec{u}(\vec{r})$
  - motion of mass element $\Delta m = \rho \cdot \Delta V$ described via forces acting on volume element
    - –pressure difference $\vec{F}_p = -\vec{\nabla} p dV$
    - –gravity $\vec{F}_g = \vec{g} \rho dV$
    - –friction $\vec{F}_R = \eta \Delta \vec{u} dV$
  - continuity equation $d\rho / dt + div(\rho \vec{u}) = 0$

Hydro in AA collisions

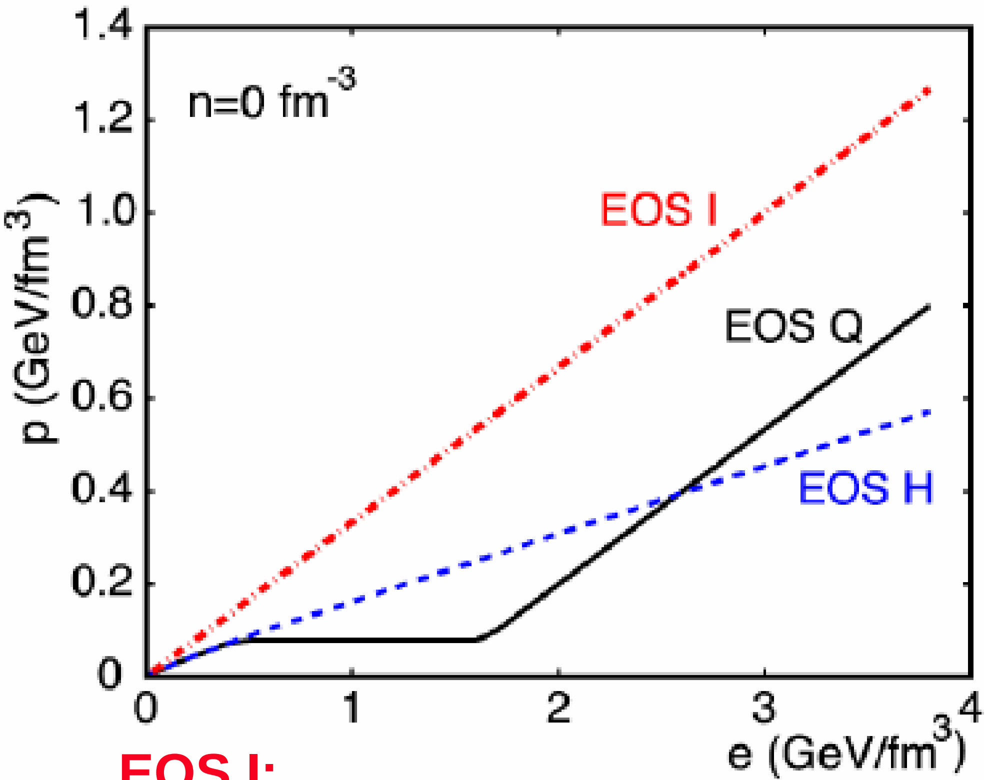
- basic approach
  - evolution of fireball = hydrodynamical evolution of an ideal fluid
    - early thermalization
    - mean free path $\lambda = 0$
    - viscosity $\eta = 0$
- necessary input
  - equation of state, relating thermodynamical variables with each other
    - $p(\varepsilon, ...)$
  - take equation of state for quark-gluon plasma from lattice QCD (for example)

Hydro in AA collisions

- **ingredients of hydro calculations**
  - **equation of motion + baryon number conservation**

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu j_B^\mu(x) = 0$$

    - 5 equations with 6 unknowns (u_x, u_y, u_z, ε, p, n_B)
  - **equation of state (EOS) p(ε, n_B)**
  - **initial conditions, e.g. from Glauber model**
  - **freeze-out condition, e.g. limit for local energy density**



EOS I:
ultra relativistic gas, p = 1/3 ε
EOS H:
resonance gas, p ≈ 0,15 ε
EOS Q:
phase transition resonance gas ↔ QGP

Hydro: "it ain't so easy"

Shen, Heinz, Huovinen, Song, arXiv:1010.1856

We used the following analytic parametrization for the equation of state s95p-PCE (energy density e and pressure p in GeV/fm^3, entropy density s in fm^{-3}, temperature T in GeV):

1. Pressure:

$$p(e) = \begin{cases} 0.3299\,[\exp(0.4346e) - 1] & : \; e < e_1 \\ 1.024{\cdot}10^{-7} \cdot \exp(6.041e) + 0.007273 + 0.14578e & : \; e_1 < e < e_2 \\ 0.30195\exp(0.31308e) - 0.256232 & : \; e_2 < e < e_3 \\ 0.332e - 0.3223e^{0.4585} - 0.003906e{\cdot}\exp(-0.05697e) + 0.1167e^{-1.233} + 0.1436e{\cdot}\exp(-0.9131e) & : \; e_3 < e < e_4 \\ 0.3327e - 0.3223e^{0.4585} - 0.003906e{\cdot}\exp(-0.05697e) & : \; e > e_4 \end{cases} \quad (1)$$

where $e_1 = 0.5028563305441270$ GeV/fm^3, $e_2 = 1.62$ GeV/fm^3, $e_3 = 1.86$ GeV/fm^3, and $e_4 = 9.9878355786273545$ GeV/fm^3.

2. Entropy density:

$$s^{\frac{4}{3}}(e) = \begin{cases} 12.2304e^{1.16849} & : \; e < e_1 \\ 11.9279e^{1.15635} & : \; e_1 < e < e_2 \\ 0.0580578 + 11.833e^{1.16187} & : \; e_2 < e < e_3 \\ \left.\begin{array}{l} 18.202e - 62.021814 - 4.85479\exp(-2.72407{\cdot}10^{-11}e^{4.54886}) \\ \quad +65.1272e^{-0.128012}\exp(-0.00369624e^{1.18735}) - 4.75253e^{-1.18423} \end{array}\right\} & : \; e_3 < e < e_4 \\ \left.\begin{array}{l} 18.202e - 63.0218 - 4.85479\exp(-2.72407{\cdot}10^{-11}e^{4.54886}) \\ \quad +65.1272e^{-0.128012}\exp(-0.00369624e^{1.18735}) \end{array}\right\} & : \; e > e_4 \end{cases} \quad (2)$$

where $e_1 = 0.1270769021427449$ GeV/fm^3, $e_2 = 0.4467079524674040$ GeV/fm^3, $e_3 = 1.9402832534193788$ GeV/fm^3, and $e_4 = 3.7292474570977285$ GeV/fm^3.

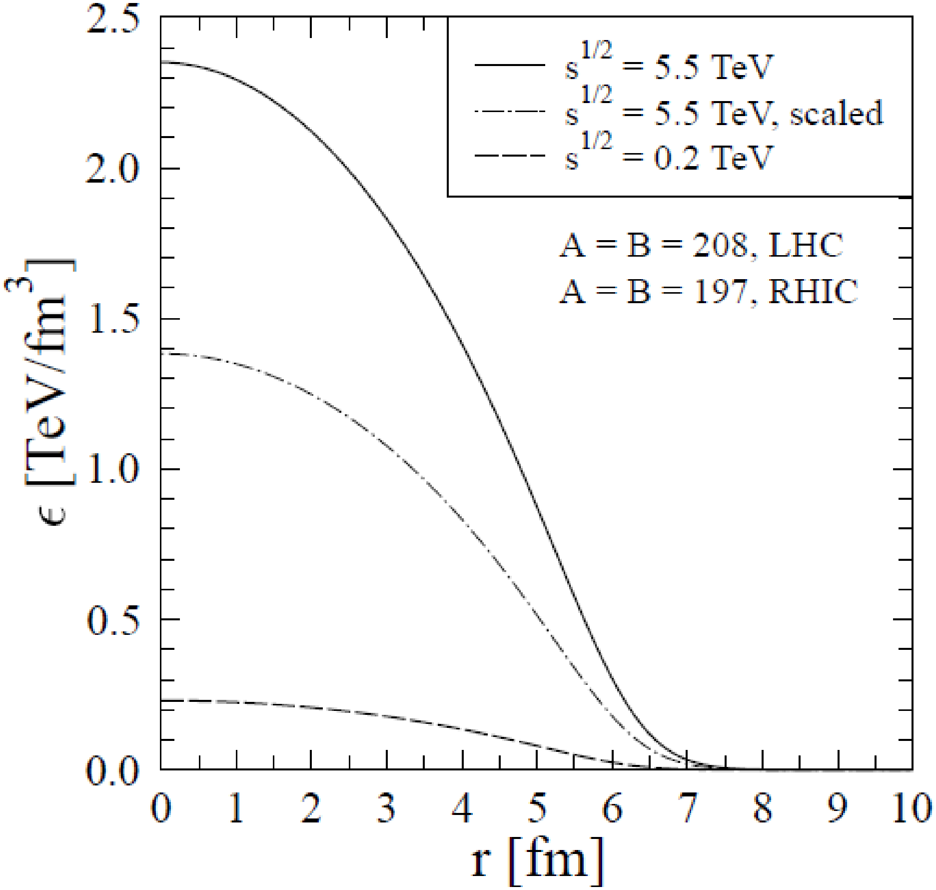
3. Temperature:

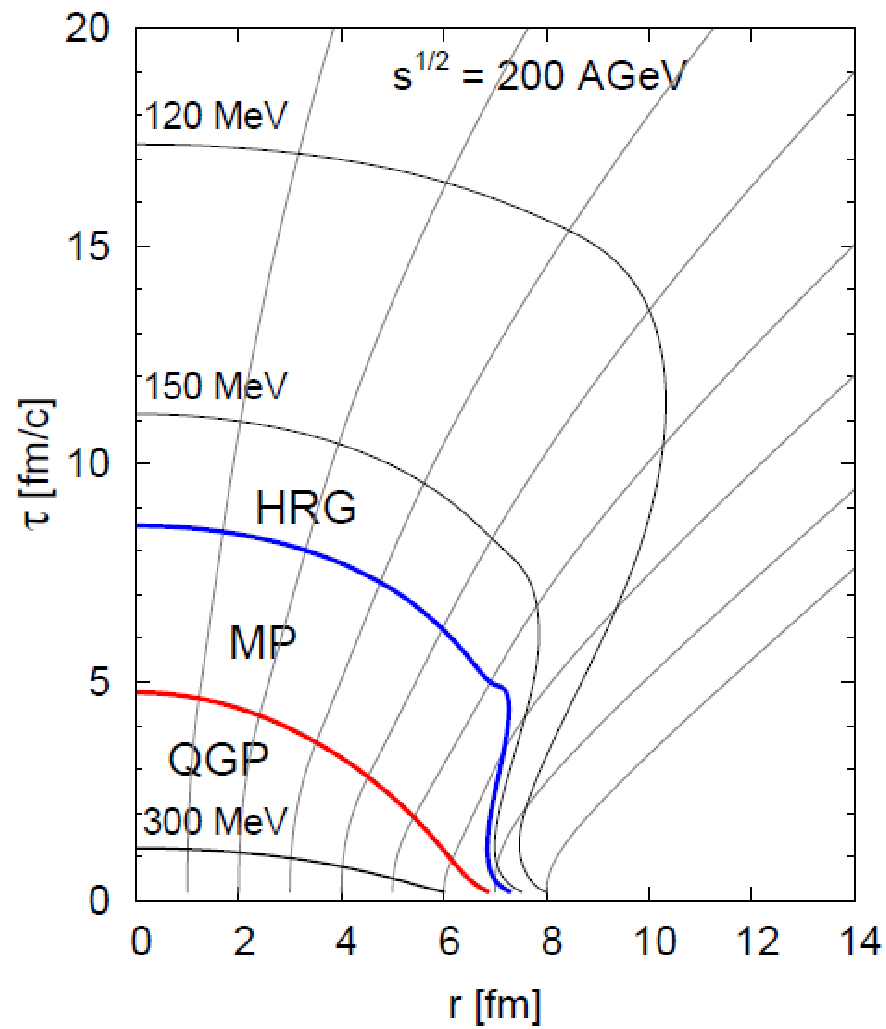
$$T(e) = \begin{cases} 0.203054e^{0.30679} & : \; e < 0.5143939846236409\,\mathrm{GeV/fm^3} \\ (e+p)/s & : \; e > 0.5143939846236409\,\mathrm{GeV/fm^3} \end{cases} \quad (3)$$

Hydro calculations

- **initial conditions and time evolution**

review: Huovinen, Ruuskanen, arXiv:nucl-th/0605008

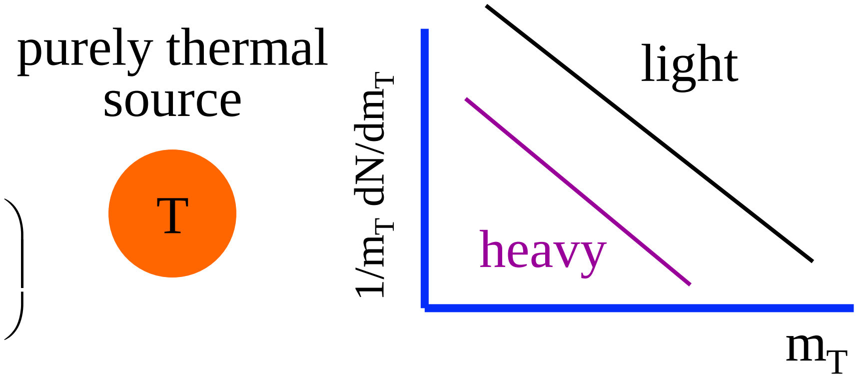




„Thermal“ momentum spectra

- **particle emission from a thermal source**

$$E\frac{d^3N}{d^3p} = \frac{gV}{(2\pi)^3} E e^{-(E-\mu)/T}$$



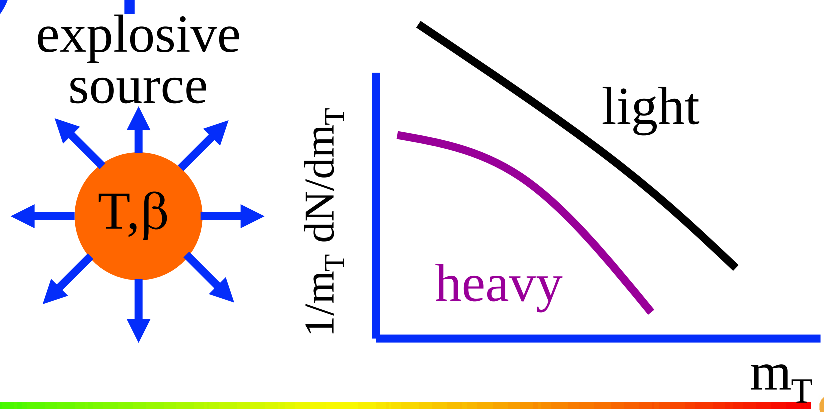
  - **m_T distribution**

$$E\frac{d^3N}{d^3p} = \frac{1}{2\pi m_T}\frac{d^2N}{dm_T\,dy} = C m_T K_1\left(\frac{m_T}{T}\right)$$

$$\approx C'\sqrt{m_T}\, e^{-m_T/T} \quad \text{für} \quad m_T >> T$$

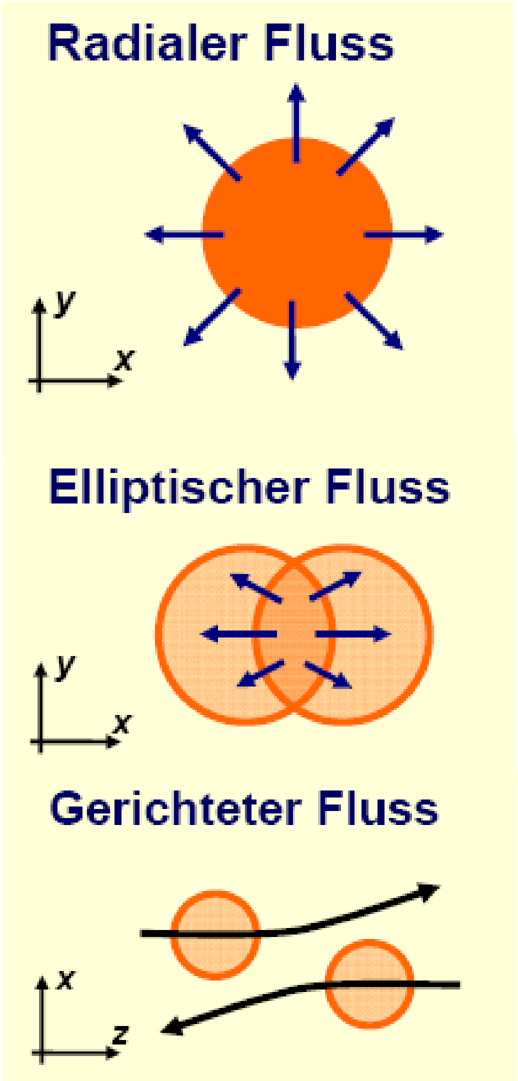
→ **m_T spectra have the same shape for all particles**

- **additional radial (collective) expansion**
  - **spectral shape different for particles with different mass**
  - **main parameter: expansion velocity β**



Collective flow of particles

- **different kinds of flow**
  - **radial flow**
    - transverse flow even for collisions with b = 0
    - leaves its footprint on p_t spectra of emitted partics
  - **elliptic flow**
    - related to initial spatial anisotropy of collision zone (b ≠ 0)
    - necessary: early thermalization
  - **directed flow**
    - related to pre-equilibrium phase of the collision
    - decreases with increasing collision energy

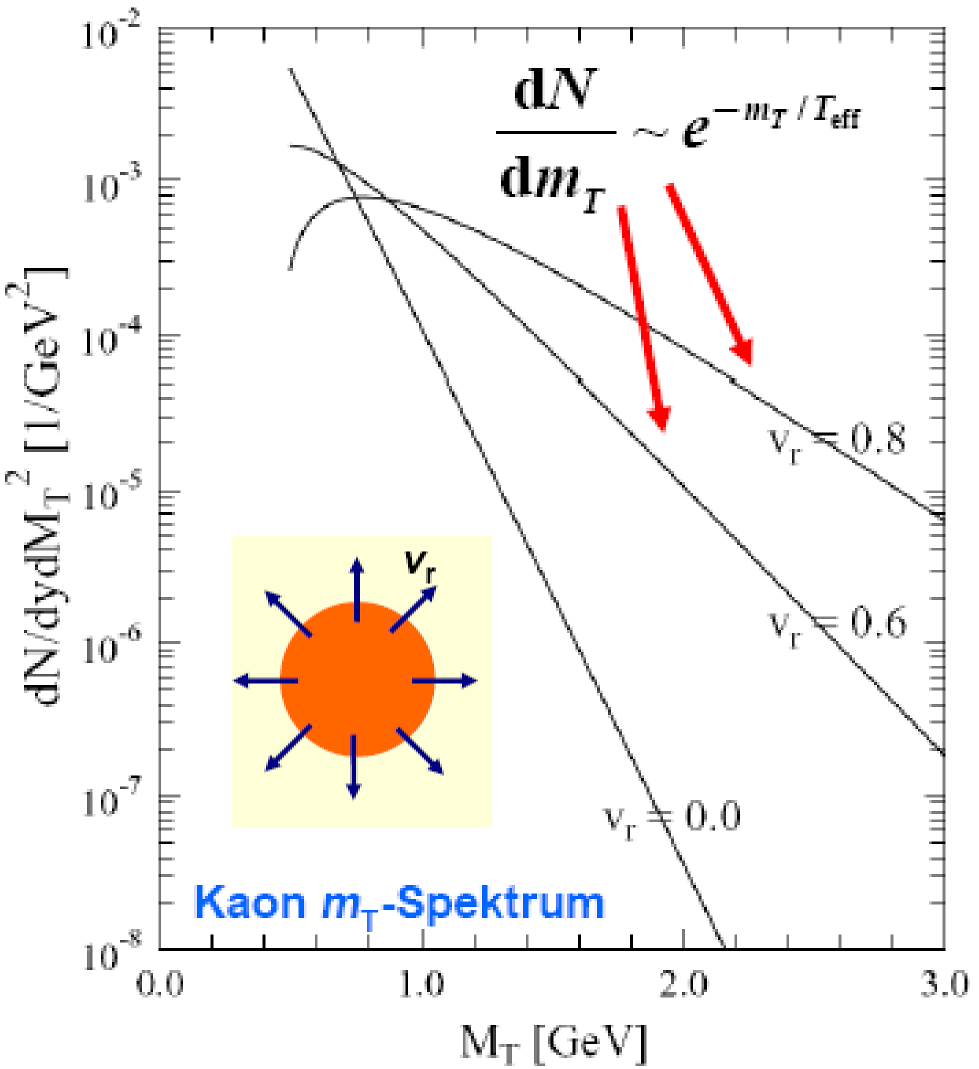


Radial flow

- radial expansion velocity v_r modifies the m_T spectra dependent on the particle mass m
- inverse slope of m_T spectra depends on v_r

$$T_{eff} = T\sqrt{\frac{1+v_r}{1-v_r}}$$

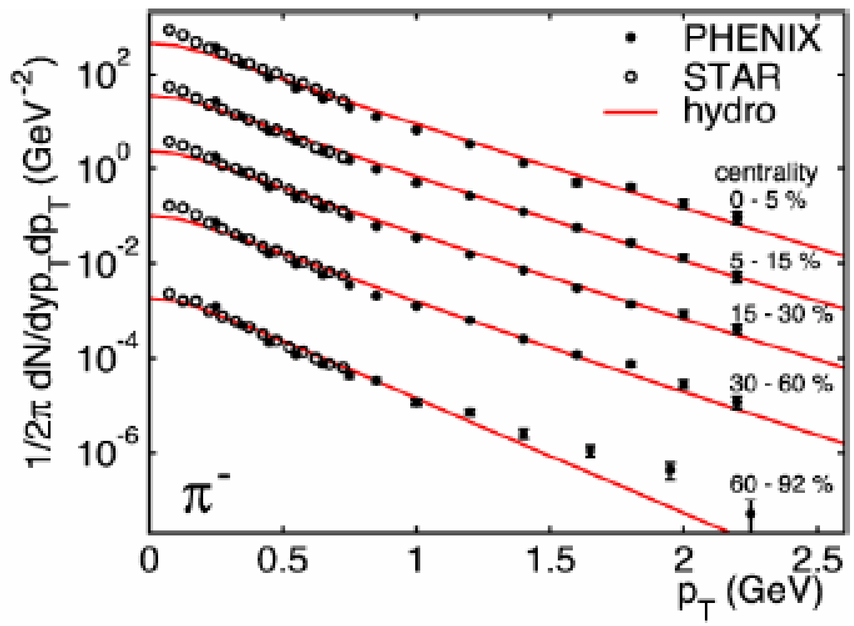
review: Huovinen, Ruuskanen, arXiv:nucl-th/0605008



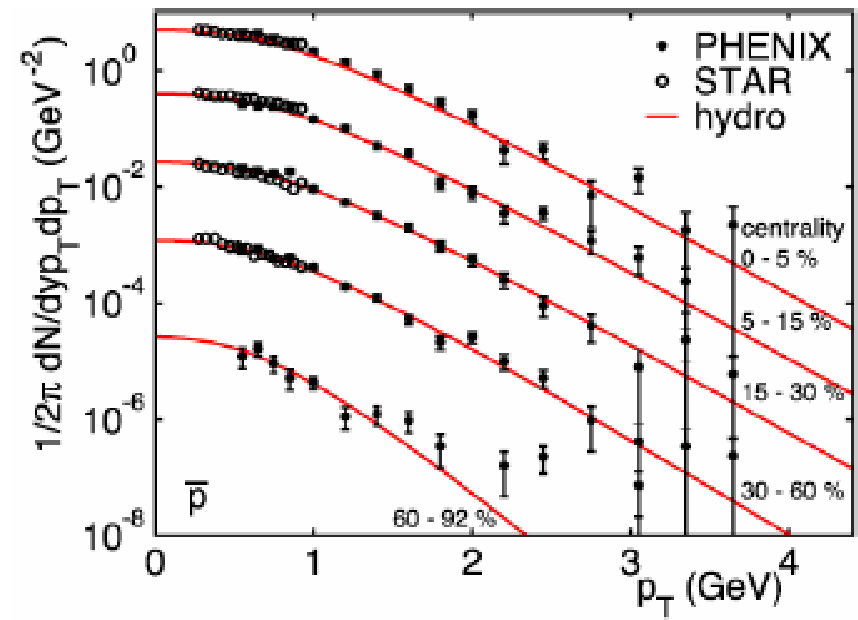
Particle spectra

- hydrodynamical calculations describe particle spectra over a large centrality range at RHIC (up to b ≈ 9 fm)
- large transverse expansion velocity at RHIC (v_r > 0.5 c)

Negative Pionen



Anti-Protonen

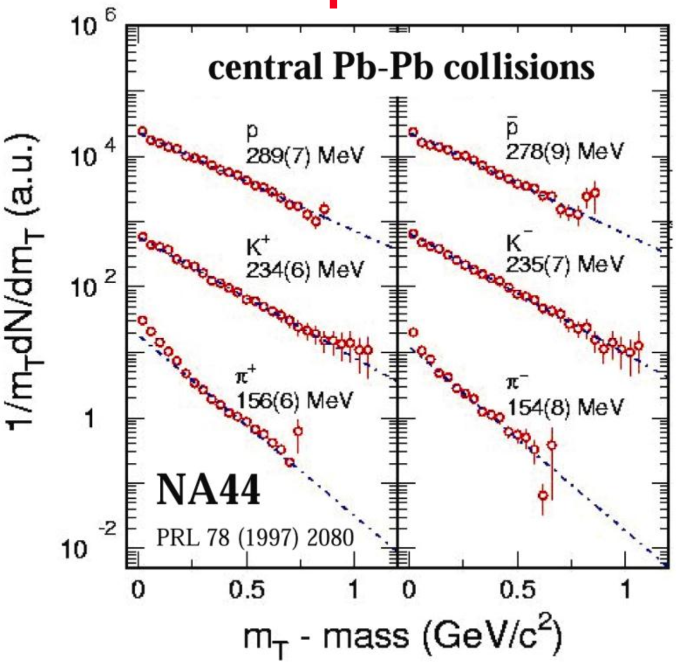


Thermal spectra + flow

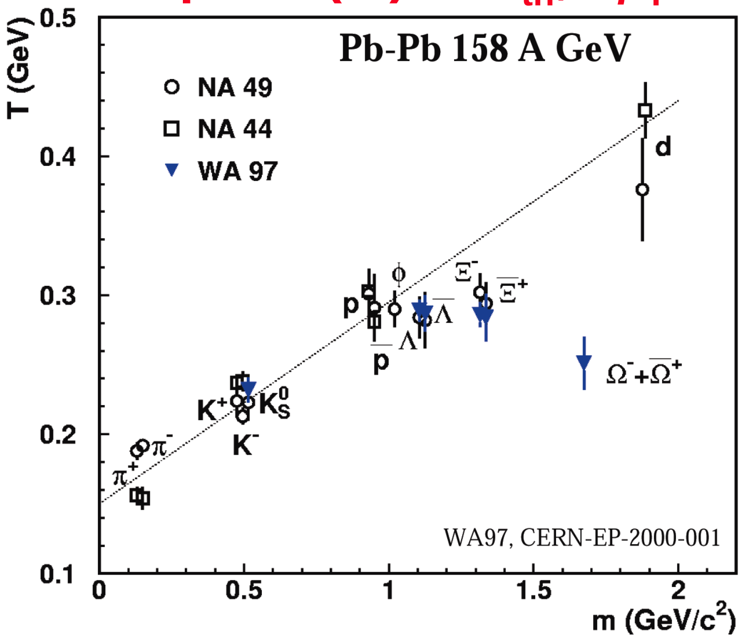
- „traditional“ Ansatz: same flow profile ($\beta(r)$) and temperature T_{th} for all particles

$$T = \begin{cases} T_{th} + \frac{1}{2} m \langle \beta_T \rangle^2 & \text{für} \quad p_T \le m \\ T_{th} \sqrt{\dfrac{1 + \langle \beta_T \rangle}{1 - \langle \beta_T \rangle}} & \text{für} \quad p_T >> m \end{cases}$$

1. fit spectra → T



2. plot T(m) → T_{th}, $\langle\beta_T\rangle$



Thermal spectra + flow

- „Blastwave“: based on hydrodynamics
- spectrum of an expanding thermal source

$$\frac{dN}{m_T\, dm_T} \propto \int_0^R r\,dr\,m_T\, I_0\left(\frac{p_T \sinh \rho}{T}\right) K_1\left(\frac{m_T \cosh \rho}{T}\right)$$

- **with transverse velocity profile**

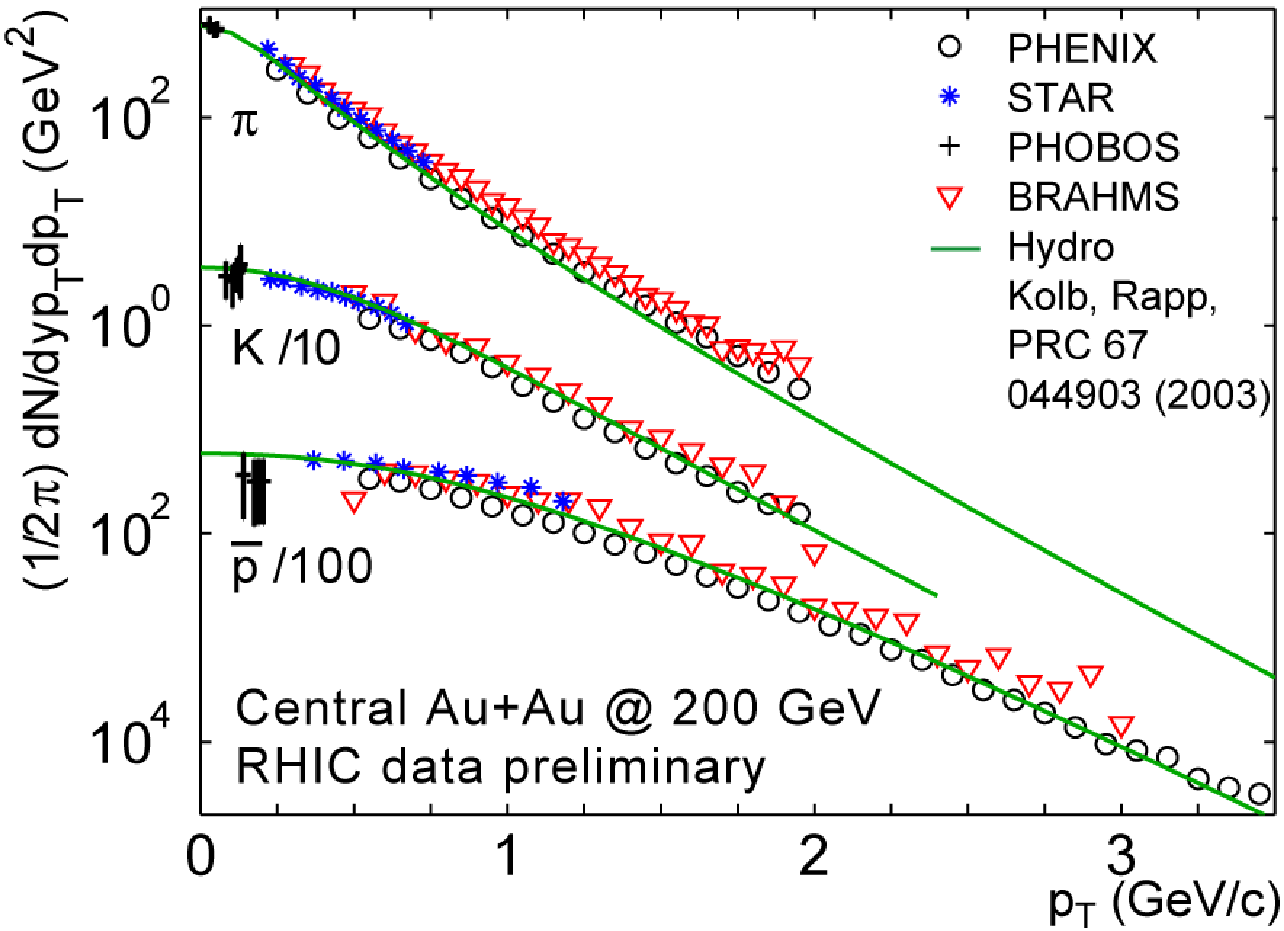
$$\beta_r(r) = \beta_s \left(\frac{r}{R}\right)^n$$

- **and rapidity boost**

$$\rho = \tanh^{-1} \beta_r$$

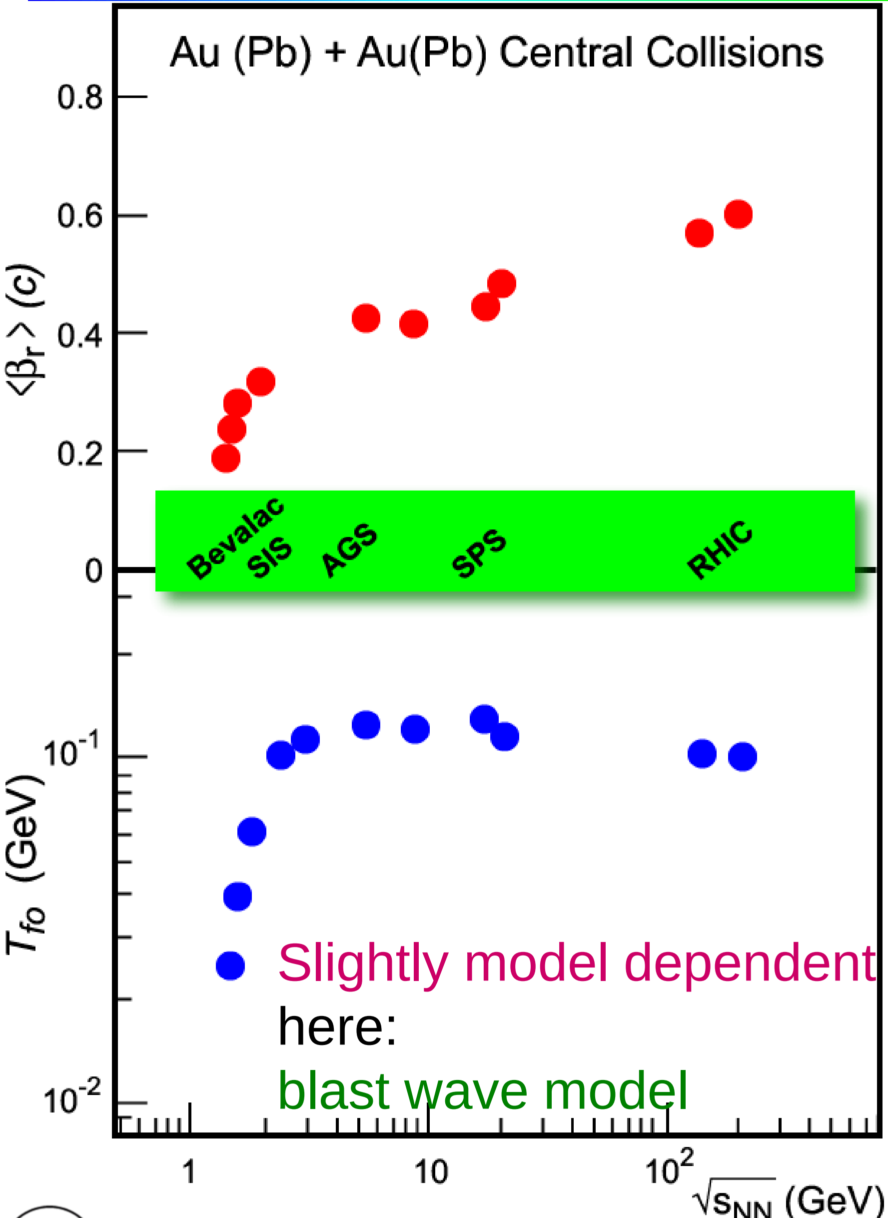
(E. Schnedermann, J. Sollfrank, U. Heinz, Phys.Rev. C48(1993)2462)

Blastwave fits at RHIC



- **good description of identified hadron spectra**
  - **T ~ 100 MeV**
  - **$\langle\beta_r\rangle$ ~ 0.55 c**

Kinetic freeze-out systematics

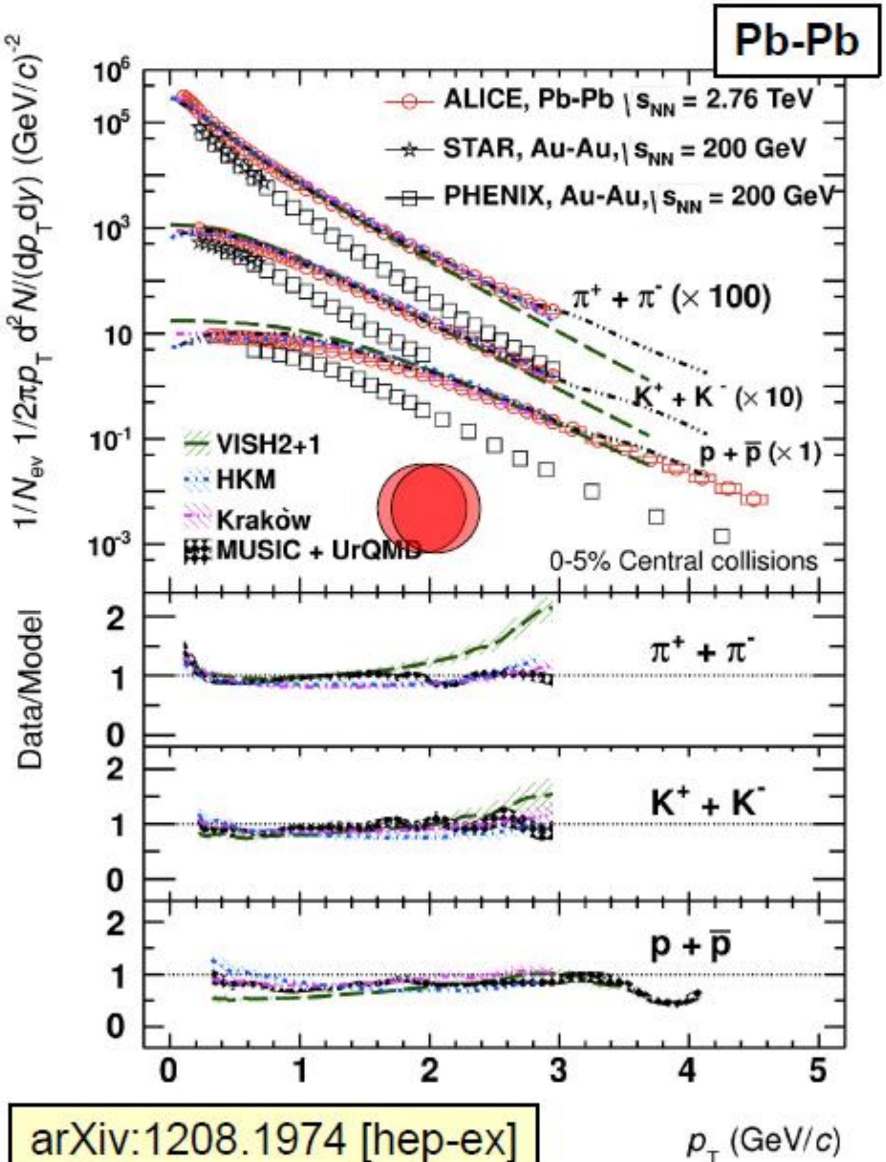


- Slightly model dependent here: blast wave model

- $<\beta_r>$
  - grows with $\sqrt{s_{NN}}$
- T
  - grows steeply with $\sqrt{s_{NN}}$ but saturates at AGS energies

strong collective radial expansion at RHIC and the LHC
⇒ large pressue
⇒ large scattering rates
⇒ thermalization is *likely!*

Blast wave fits at the LHC



arXiv:1208.1974 [hep-ex]

p_T spectra → from thermal sources expanding with a collective transverse radial flow velocity β_T

Fit to the data with Blast-Wave model
Schnedermann et al., PRC 48, 2462 (1993)

$\langle\beta_T\rangle = 0.65 \pm 0.02$

~10% higher than at RHIC

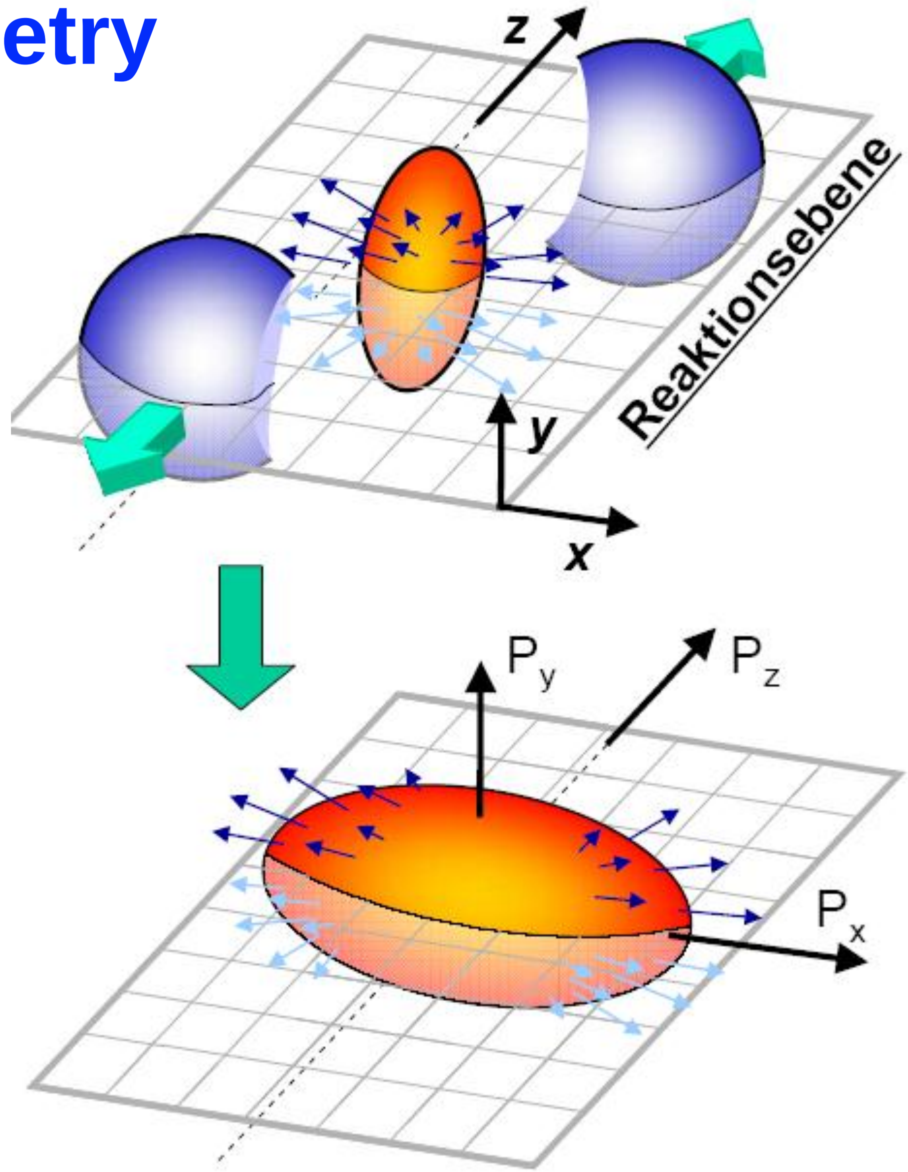
$T_{kin} = 96 \pm 10$ MeV

compatible within errors

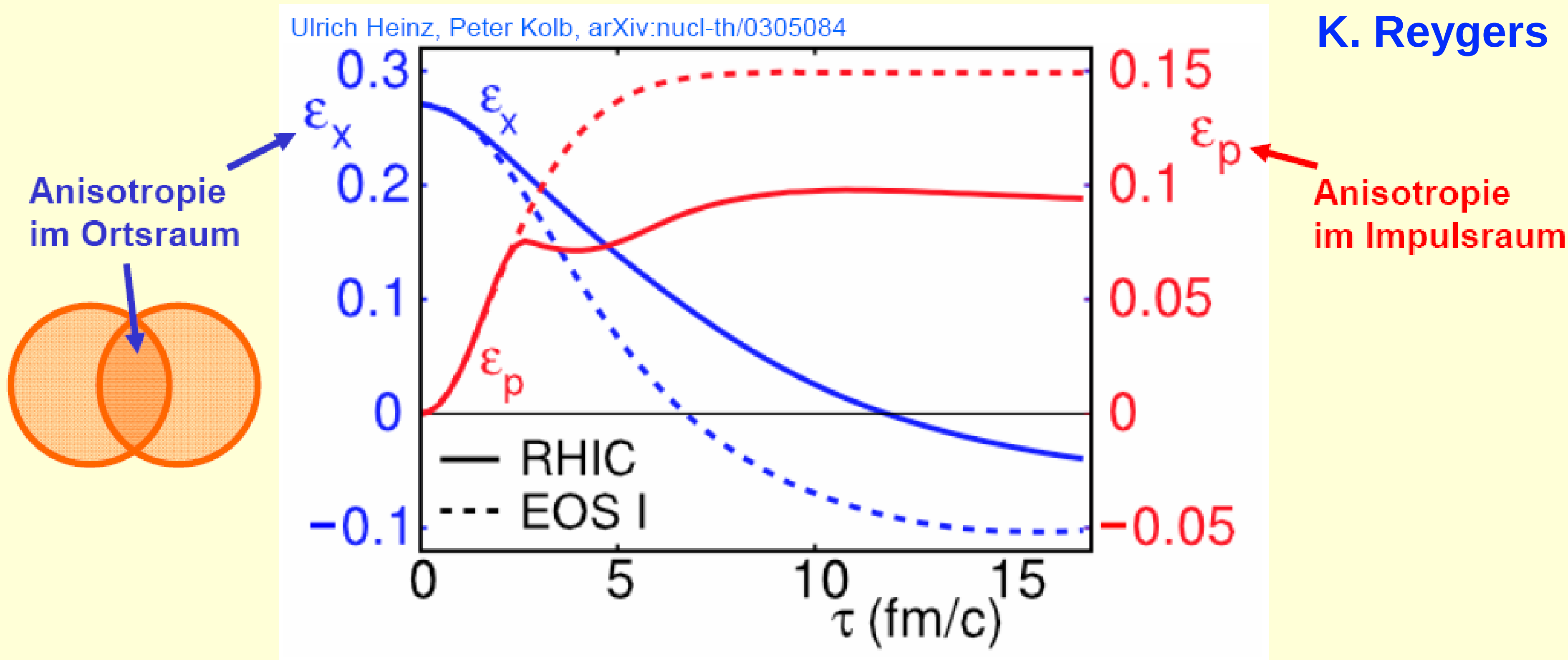
- consistent with previous systematics!

Elliptic flow

- initial spatial asymmetry of overlap zone
  → anisotropy of momentum distr. w.r.t. the orientation of the reaction plane
  - origin
    - asymmetric pressure gradients
  - prerequisites
    - early thermalization
    - strong „coupling“
  - „self quenching“



Time evolution of elliptic flow

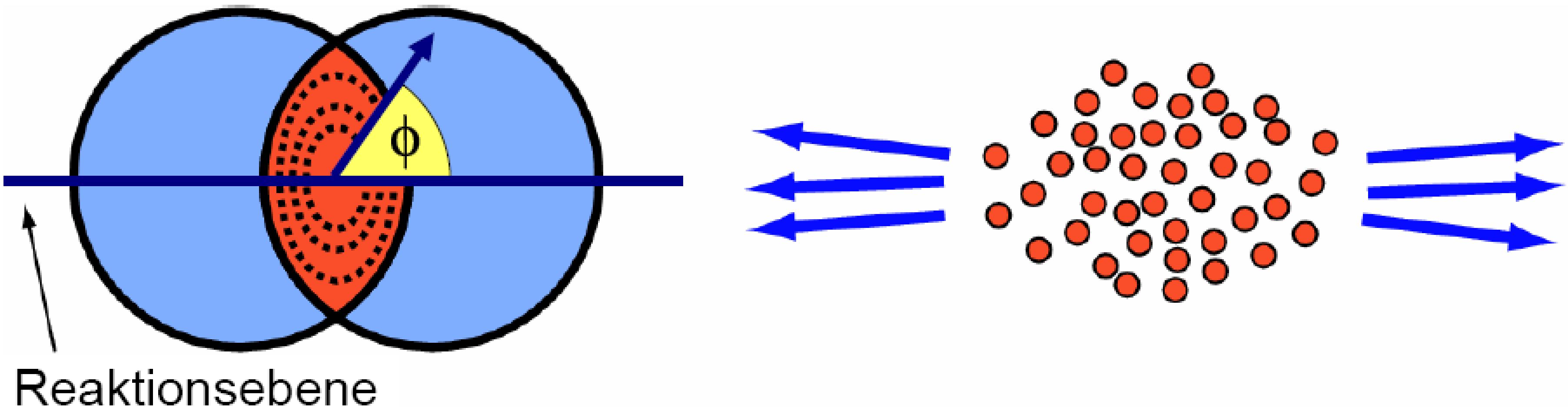


K. Reygers

- **hydrodynamic models: elliptic flow develops in the early phase of the fireball evolution (quark-gluon plasma)**

Fourier expansion

Anisotropie im Ortsraum ⇒ Anisotropie im Impulsraum



- **Fourier expansion of particle distribution**

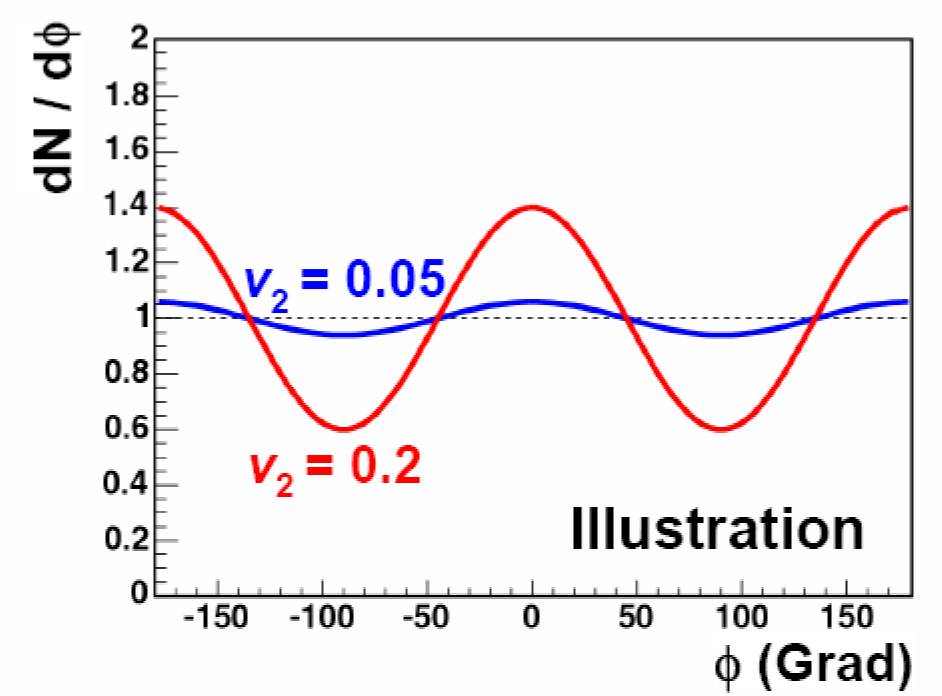
$$\frac{d^2N}{d\phi dp_T} = N_0 \cdot \left(1 + 2v_1(p_T)\cos(\phi) + 2v_2(p_T)\cos(2\phi) + \dots\right)$$

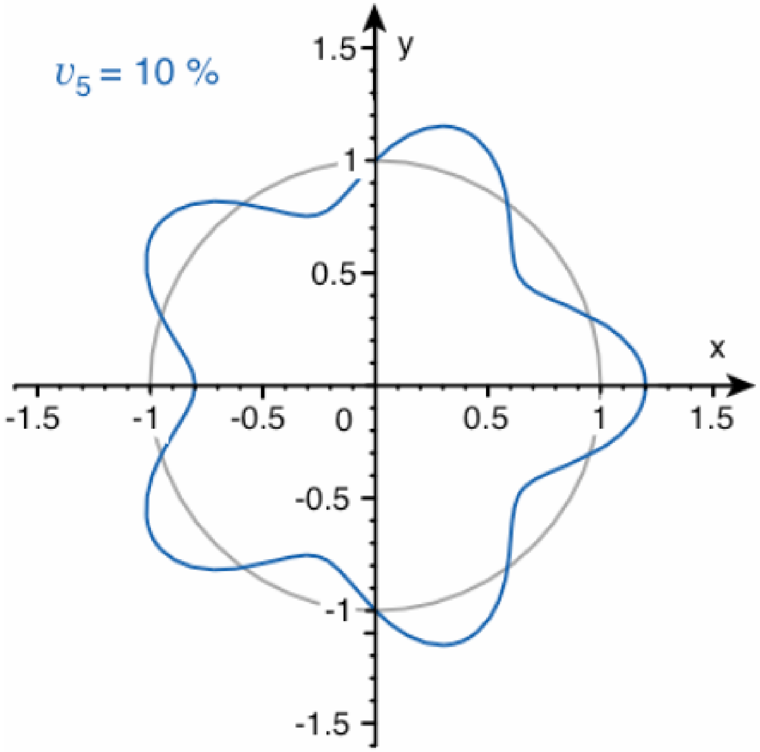
- $\phi = \phi_{particle} - \phi_{reaction\ plane}$
- v_1: directed flow strength (0 at y_{CMS})
- v_2: elliptic flow strength

Fourier expansion

$$\frac{dN_i}{dyp_T\,dp_T\,d\phi} = \frac{1}{2\pi p_T}\frac{dN_i}{dydp_T}\left(1 + \sum_{n,gerade} 2v_n^i(p_T)\cos(n\phi)\right)$$

$$v_n^i(p_T) = \langle\cos(n\phi)\rangle^i = \frac{1}{dN_i/dyp_T\,dp_T}\int d\phi\cos(n\phi)\frac{dN_i}{dyp_T\,dp_T\,d\phi}$$

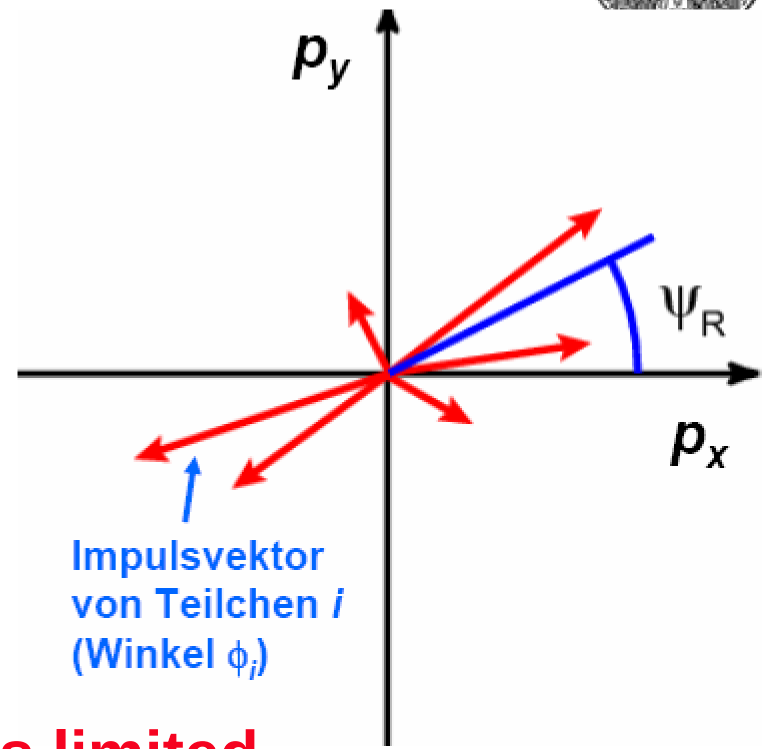




Reaction plane

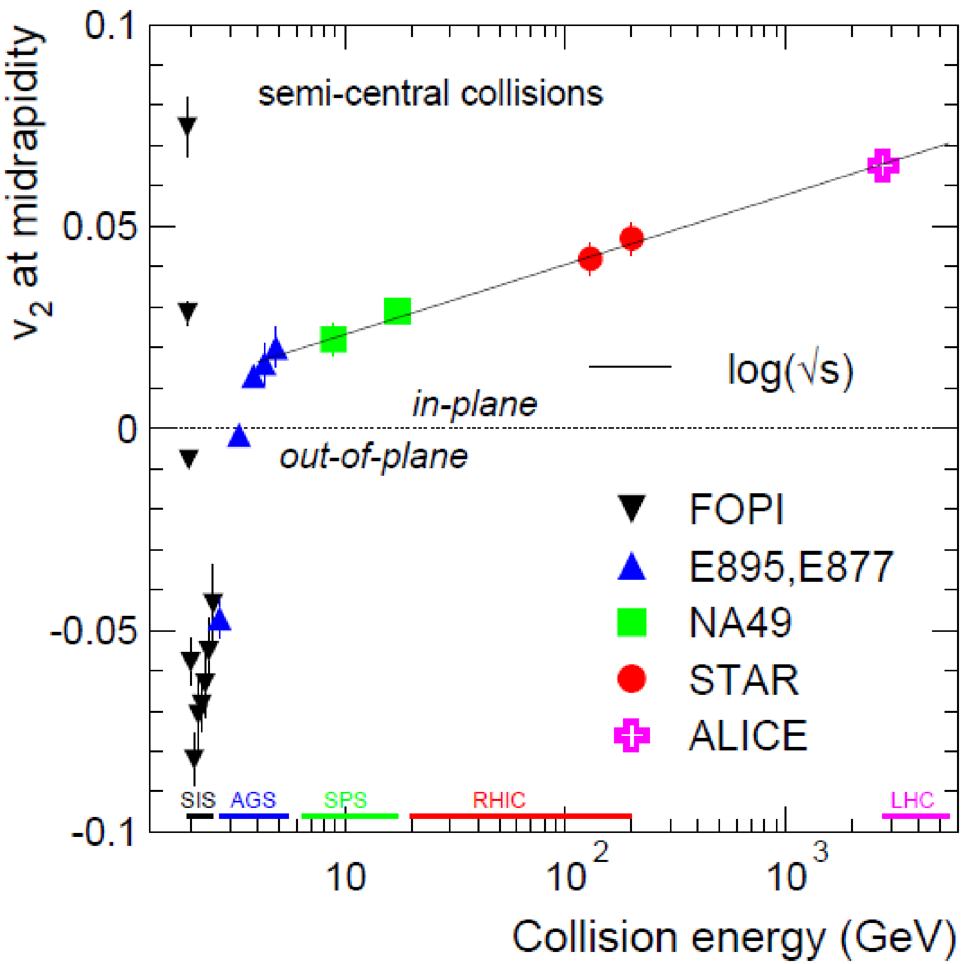
- how to measure the orientation of the reaction plane

$$\Psi_R = \frac{1}{2}\tan^{-1}\left(\frac{\sum_i w_i \cdot \sin(2\phi_i)}{\sum_i w_i \cdot \cos(2\phi_i)}\right)$$



  - choice of weights is not unique, common: $w_i = p_{T,i}$
  - resolution of the measurement is limited by the finite number of produced particles
  - resolution can be determined via „sub events“
  - measured v_2 has to be corrected for this resolution
  - „particle of interest“ is excluded from reaction plane determination

Elliptic flow: energy dependence



$v_2 > 0$ at low energies: in-plane, rotation-like emmision

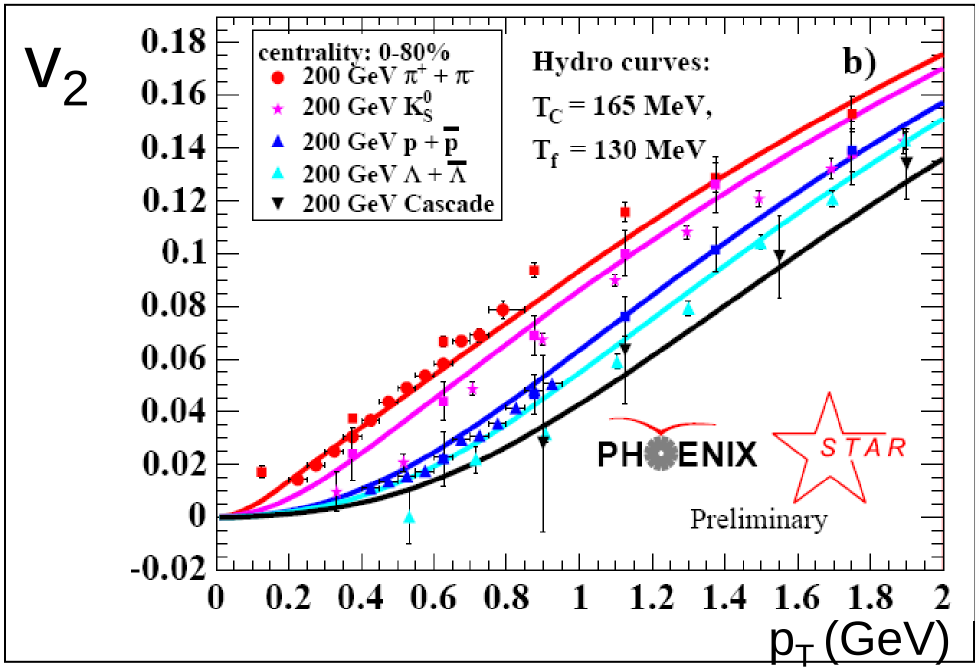
$v_2 < 0$ onset of expansion, in competition with shadowing by spectators (which act as a clock for the collective expansion, t_{pass}=40-10 fm/c)

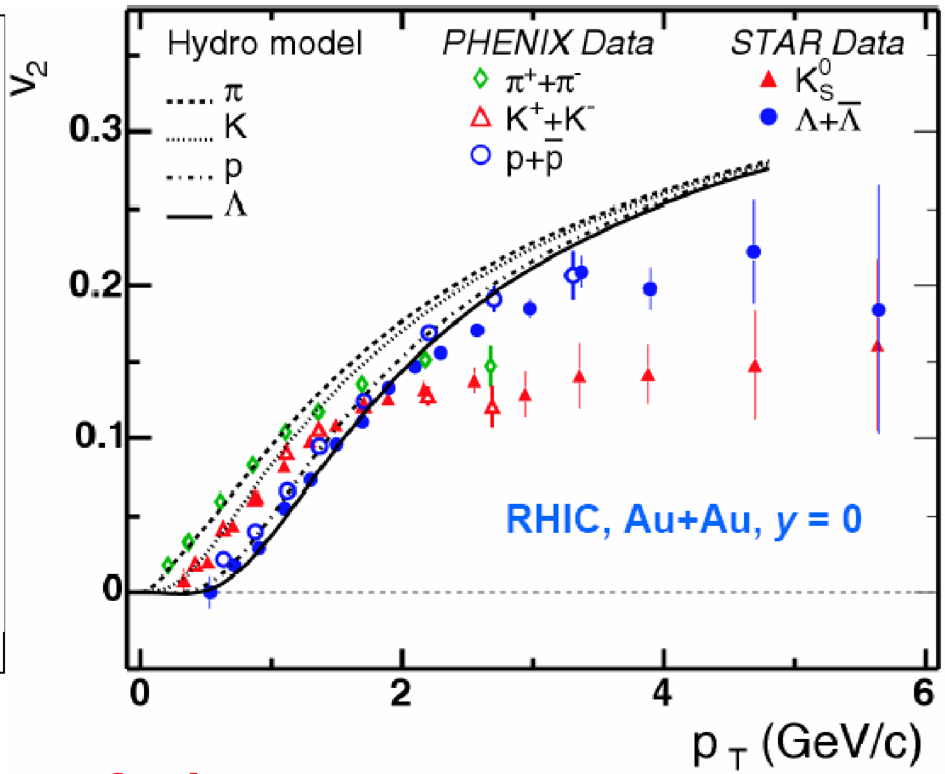
$v_2 < 0$ at high energies: "free" fireball (almond-shape) expansion ("genuine" elliptic flow)

from A. Andronic

v_2: experimental results

- „ideal“ hydro calculation describe elliptic flow data with zero viscosity!





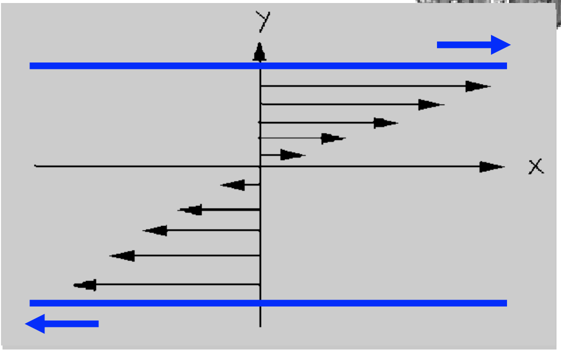
- thermalization requires $\tau < 1$ fm/c
- initial energy density: 20 GeV/fm^3

- perfect fluid (AIP "Story of the Year" 2005)

Viscosity

- shear viscosity η of a fluid
  - relation between
    - shear stress
    - velocity gradient in the fluid

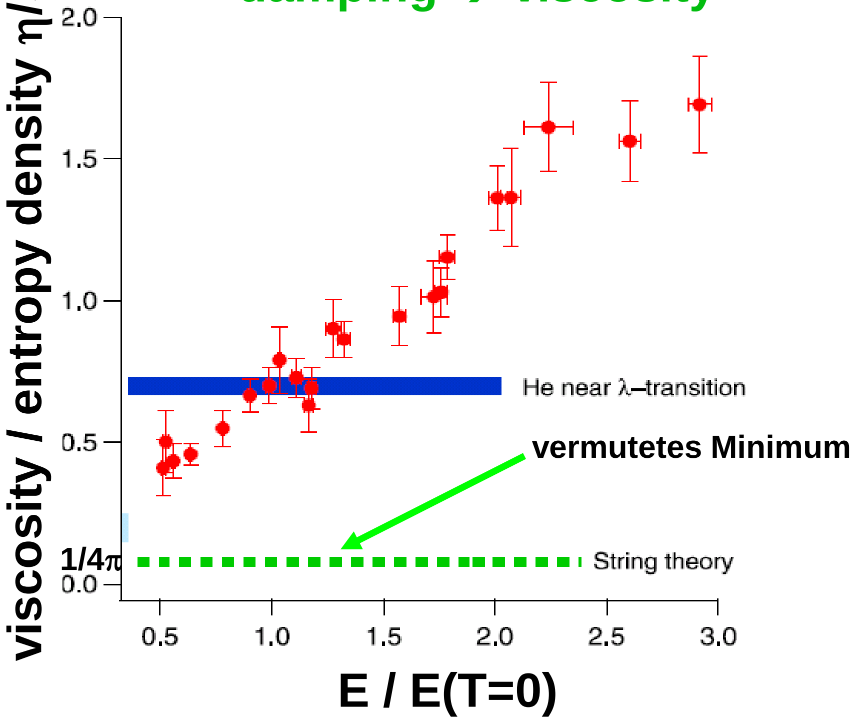
$$\frac{F_x}{A} = -\eta \frac{\partial v_x}{\partial y}$$



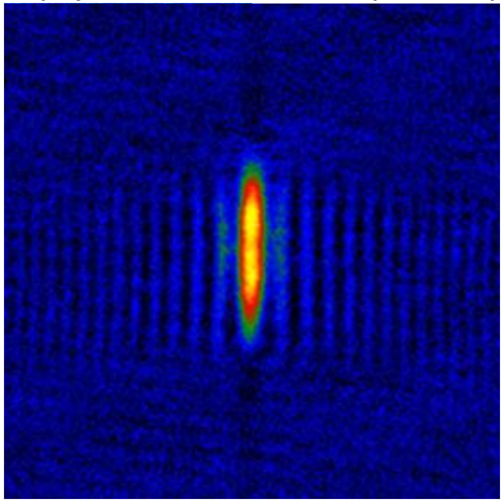
  - large interaction cross sections in the fluid ↔ low viscosity
- largest viscosity ever observed
  - pitch-drop experiment
  - lab experiment continuosly running for the longest time
  - T. Parnell: observation since 1927 (Queensland University, Australia)
  - 8 drops in 8 decades
  - eye witnesses of a drop falling: 0

Fluids with small viscosity

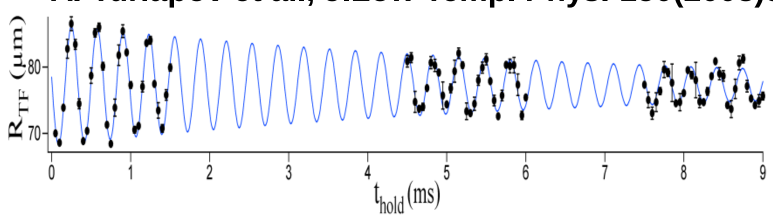
- quantum fluids
  - liquid Helium
  - strongly interacting Li atoms
    - oscillations
    - damping → viscosity



http://www.phy.duke.edu/research/photon/qoptics



A. Turlapov et al., J.Low Temp. Phys. 150(2008)567



T. Schäfer & D. Teaney, arXiv:0904.3107

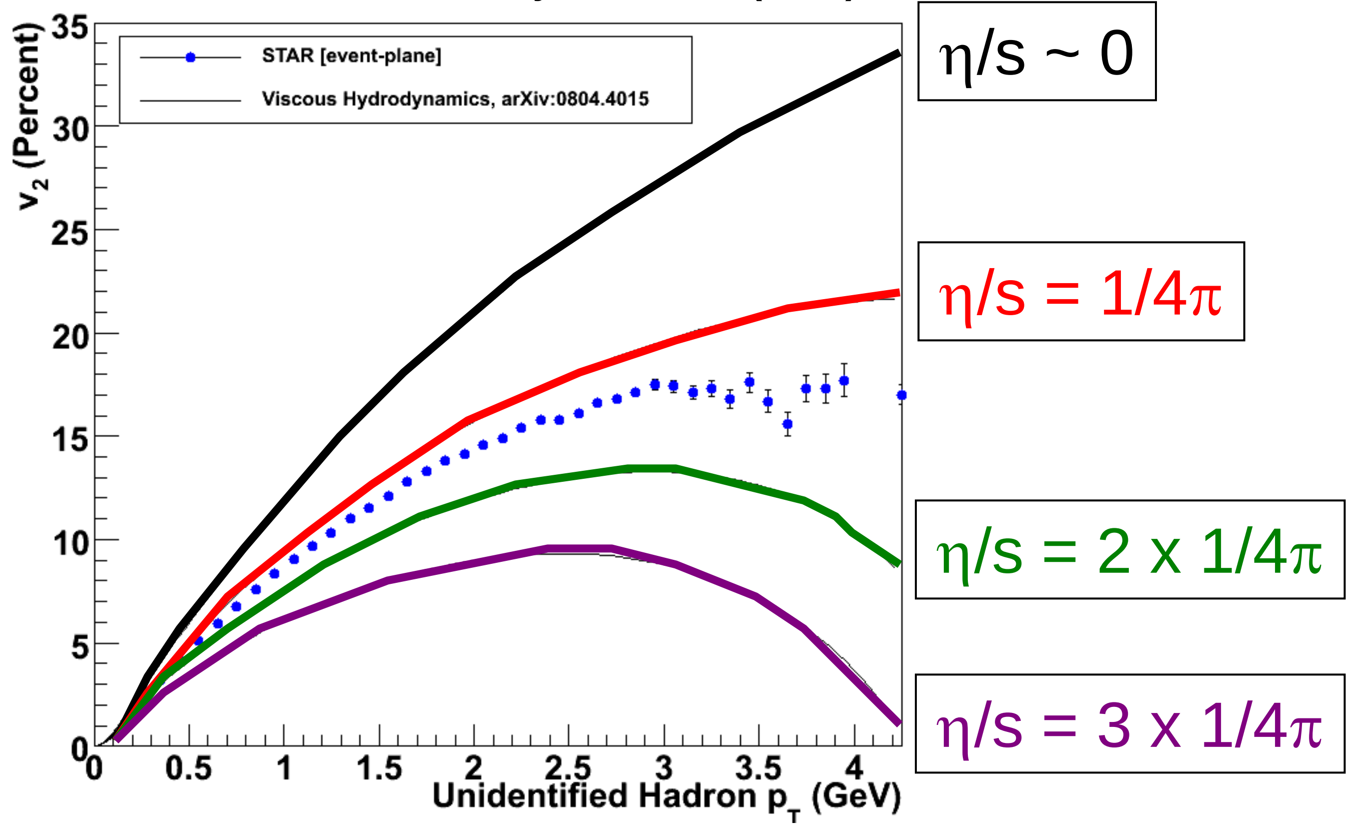
fluid	P [Pa]	T [K]	η [Pa·s]	η/n [$\hbar$]	η/s [$\hbar/k_B$]
H_2O	$0.1\cdot10^6$	370	$2.9\cdot10^{-4}$	85	8.2
^{4}He	$0.1\cdot10^6$	2.0	$1.2\cdot10^{-6}$	0.5	1.9
H_2O	$22.6\cdot10^6$	650	$6.0\cdot10^{-5}$	32	2.0
^{4}He	$0.22\cdot10^6$	5.1	$1.7\cdot10^{-6}$	1.7	0.7

- „near perfect" fluid in HIC: measurement?

η/s at RHIC?

- **compare measured elliptic flow with 3d, relativistic, viscous hydro calculations**

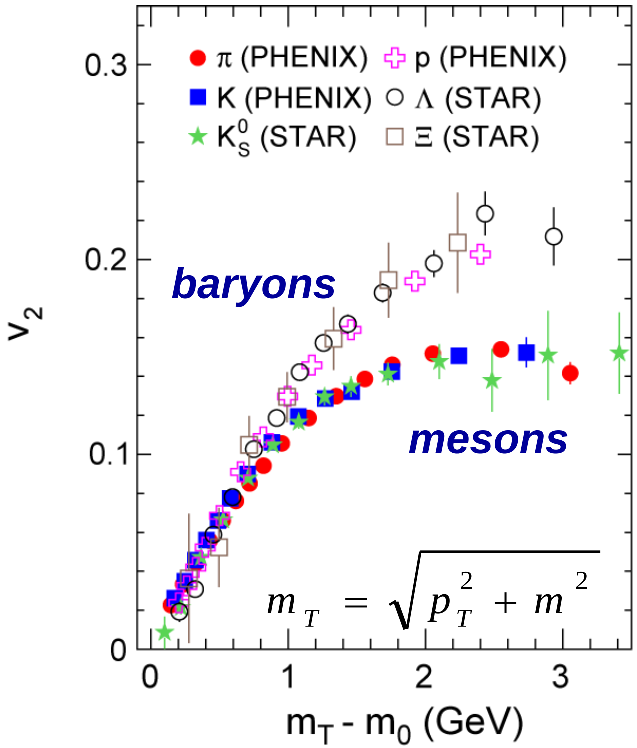
M. Luzum & P. Romatschke, Phys. Rev. C78(2008)034915



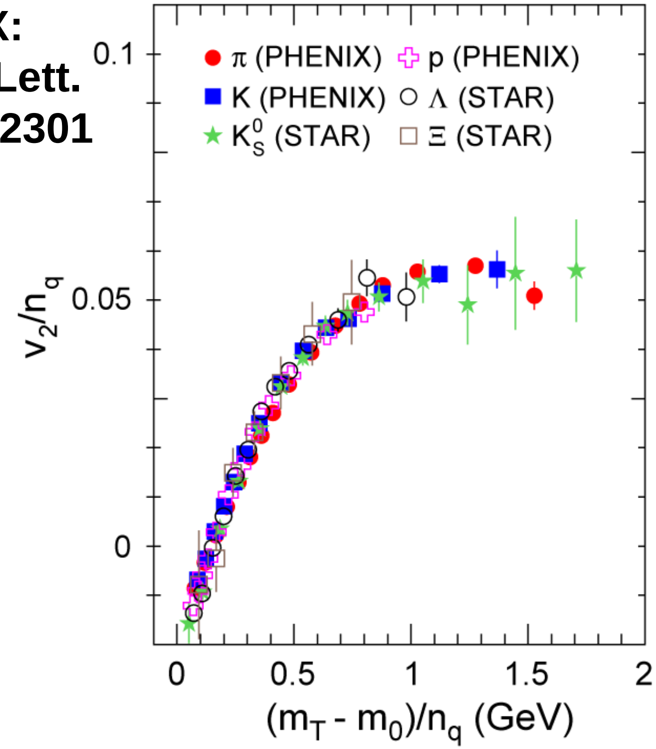
$$\left(\frac{\eta}{s}\right) \Big/ \left(\frac{1}{4\pi}\right) = 1.3 \pm 1.3 \text{ (Theorie)} \quad \pm 1.0 \text{ (Experimen} \quad \text{t)}$$

Quark scaling of elliptic flow

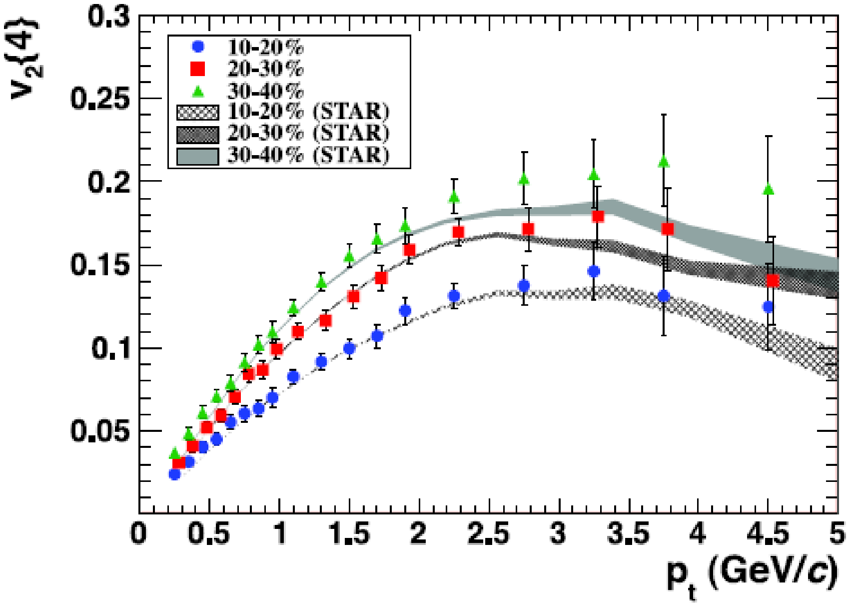
- **elliptic flow scales with quark content (no gluons?) at RHIC**
  - **violation of scaling seen at LHC and at RHIC at large $KE_T = m_T - m_0$ and in more peripheral collisions → accidental scaling??**

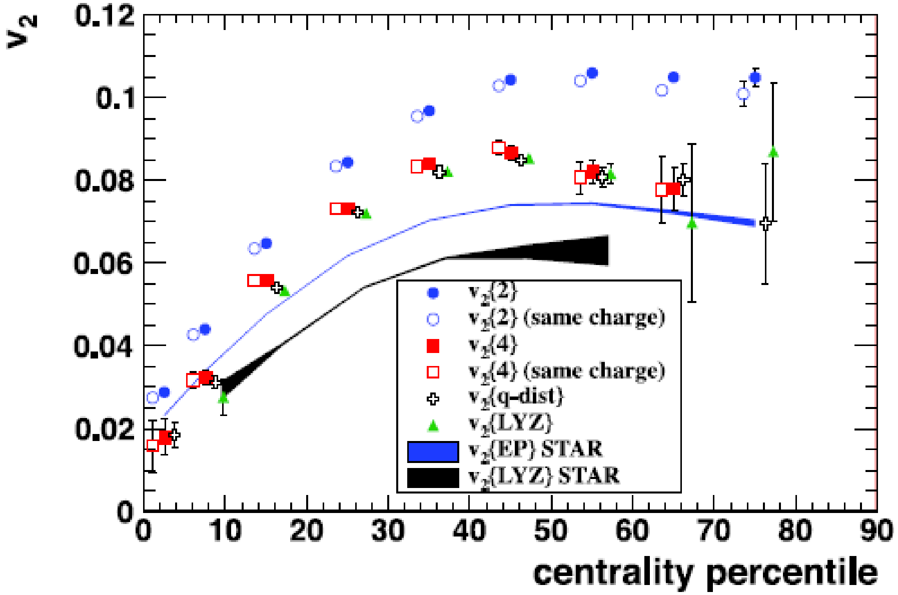


PHENIX: Phys.Rev.Lett. 98(2007)162301



Elliptic flow at the LHC





ALICE collab., arXiv:1011.3914

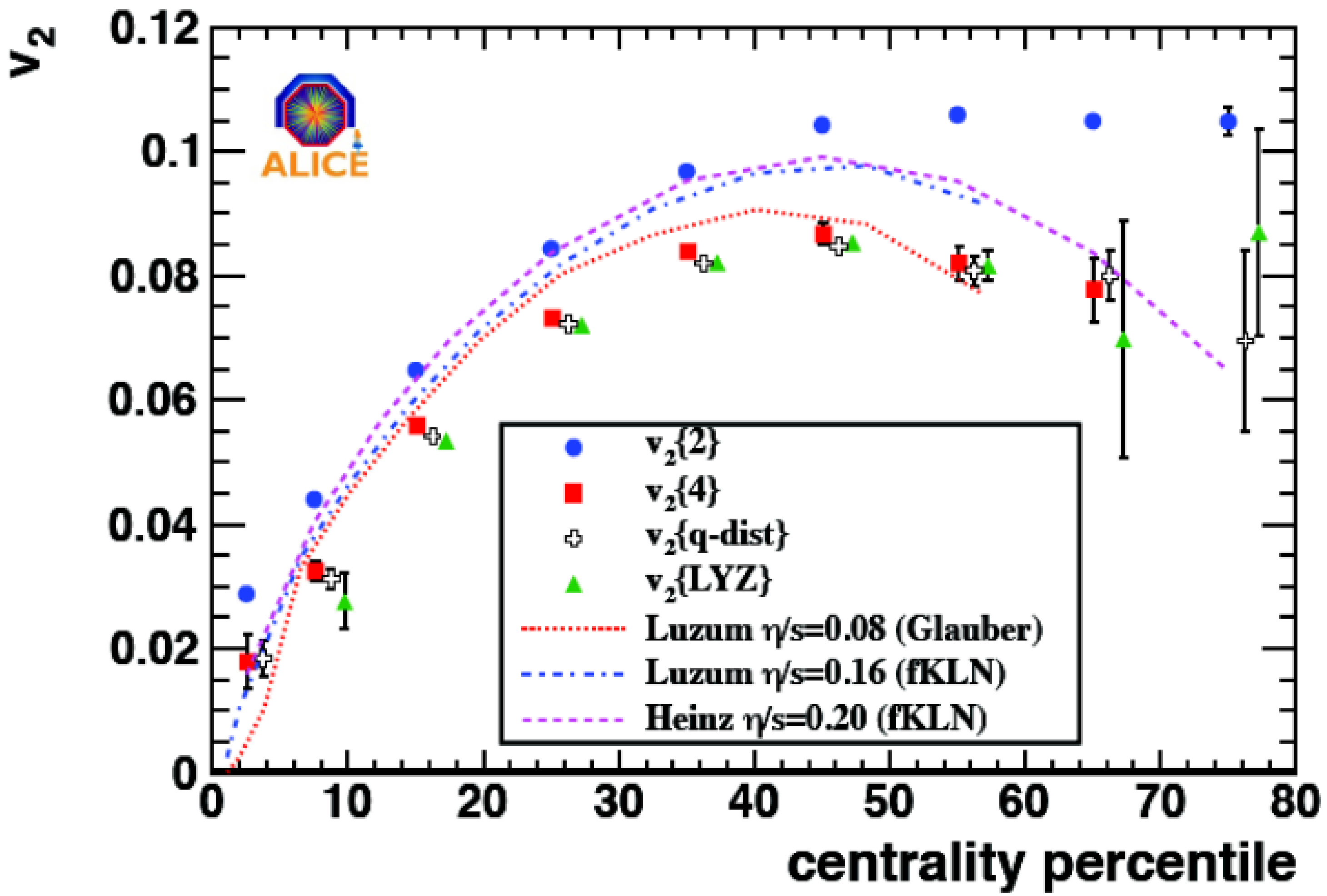
same p_t dependence as at RHIC (STAR); overall $\simeq$35% larger v_2 (larger $< p_t >$)

$v_2\{2\}$ and $v_2\{EP\}$ contain "non-flow contributions" (weak-decays, jets...)

see Bilandzic, Snellings, Voloshin, arXiv:1010.0233

from A. Andronic

Low-viscosity fluid at the LHC



Calculation:
M.Luzum,
arXiv:1011.5173

- viscous hydro calculations are in good agreement with the measured v_2 using small η/s

Hanbury-Brown Twiss interferometry

- **Hanbury-Brown and Twiss (1950s): correlation of photon intensities in independent detectors allows a measurement of the angular size of stars**
- **Goldhaber et al. (1960): study of angular correlations between pions produced in proton-antiproton collisions**
- **HBT interferometry = intensity interferometry, not amplitude interferometry (as e.g. in Young‘s double slit experiment)**

HBT formalism

Two-particle correlation function:

$$C^{12}(\mathbf{p}_1, \mathbf{p}_2) = \frac{dN^{12}/(d^3p_1 d^3p_2)}{(dN^1/d^3p_1)(dN^2/d^3p_2)} = C^{12}(\mathbf{k}, \mathbf{q}) = 1 + |\rho(\mathbf{q})|^2,$$

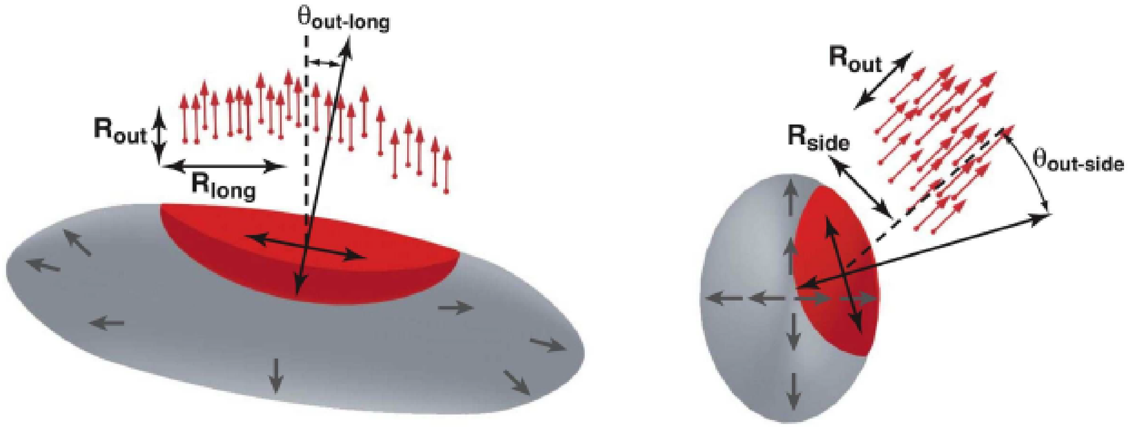
$$\mathbf{k} = (\mathbf{p}_2 + \mathbf{p}_1)/2, \ \mathbf{q} = \mathbf{p}_2 - \mathbf{p}_1$$

exhibits for bosons a (Bose-Einstein) enhancement at low-q
(due to symmetrization of w.f. ...Glauber's explanation of photon interf. HBT)

measures the distribution of (indistinguishable) particles with identical velocities

...randomly-emitted from a "region of homogeneity" (Sinyukov)

"out-side-long" ref. frame
(Bertsch-Pratt):
long (z) = beam,
out $(x) \,||\, \mathbf{k}_T$



from A. Andronic

HBT formalism

$C(\mathbf{q}) = A(\mathbf{q}) / B(\mathbf{q})$,

$A(\mathbf{q})$ measured distr. of $\mathbf{q} = \mathbf{p}_2 - \mathbf{p}_1$ of any two particles (π^-)

$B(\mathbf{q})$ same for mixed events (every event mixed with several other events)

Fitted with (central AA collisions, midrapidity; no cross terms):

$C(\mathbf{q}) = \mathcal{N}\left[(1-\lambda) + \lambda K(q_{\mathrm{inv}})(1 + G(\mathbf{q}))\right]$

R_{out}, R_{side}, R_{long} Gaussian HBT radii

$G(\mathbf{q}) = \exp(-(R_{\mathrm{out}}^2 q_{\mathrm{out}}^2 + R_{\mathrm{side}}^2 q_{\mathrm{side}}^2 + R_{\mathrm{long}}^2 q_{\mathrm{long}}^2))$

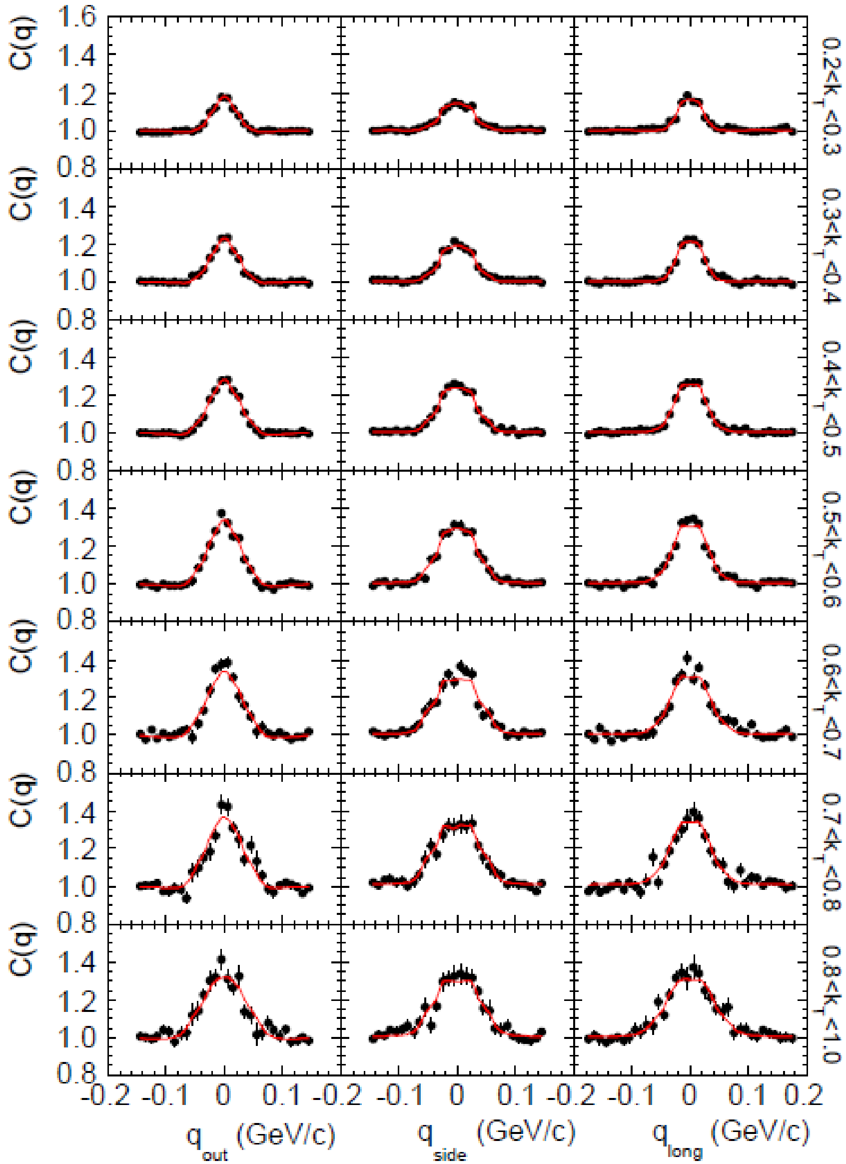
λ describes the true correlation strength; source diluted by decays or mis-id (1-λ)

$K(q_{\mathrm{inv}})$ squared Coulomb wave function averaged over a spherical source of size equal to the mean of R_{out}, R_{side}, and R_{long}

(q_{inv}, for pairs of identical pions, is equal to q calculated in the pair rest frame)

from A. Andronic

Correlation functions

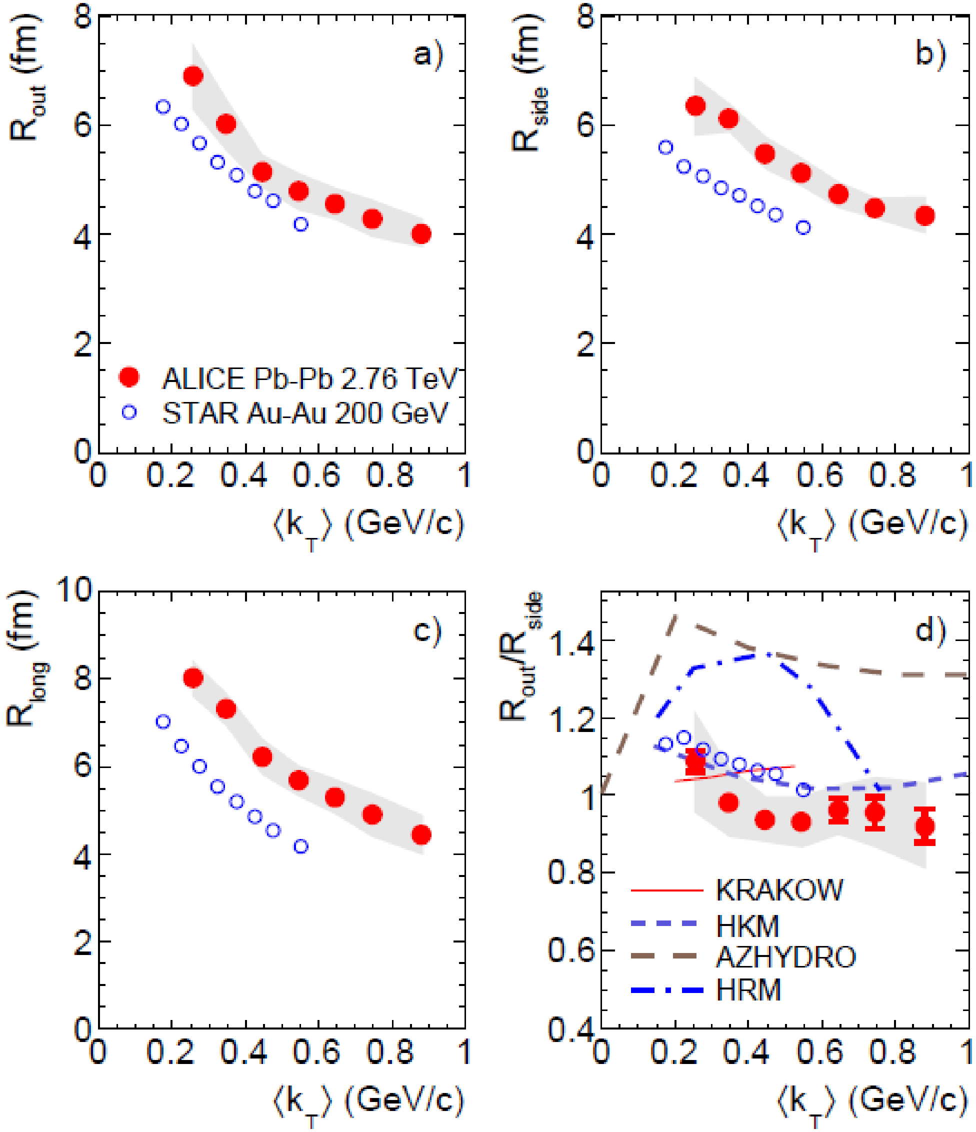


ALICE collab., arXiv:1012.4035

$\pi^-\pi^-$ ($\pi^+\pi^+$ are similar)
central Pb+Pb at LHC
$k_T = |\mathbf{p}_{T,1} + \mathbf{p}_{T,2}|/2.$

from A. Andronic

HBT radii



ALICE collab., arXiv:1012.4035

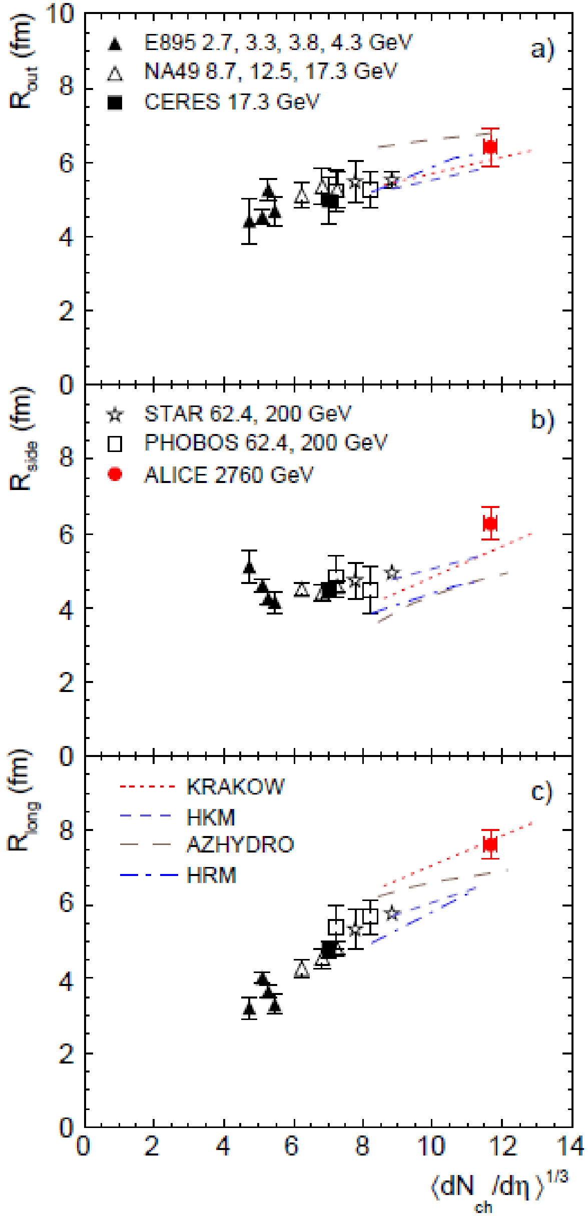
correction for momentum resolution: a few %

2 hydro models (KRAKOW, HKM) describe R_{out}/R_{side}

earlier discrepancies in this was dubbed "RHIC HBT puzzle"

from A. Andronic

Energy dependence of HBT radii



ALICE collab., arXiv:1012.4035

for k_T=0.3 GeV/c

$\langle dN_{ch}/d\eta \rangle \sim$Volume

(centrality and energy-dep.)

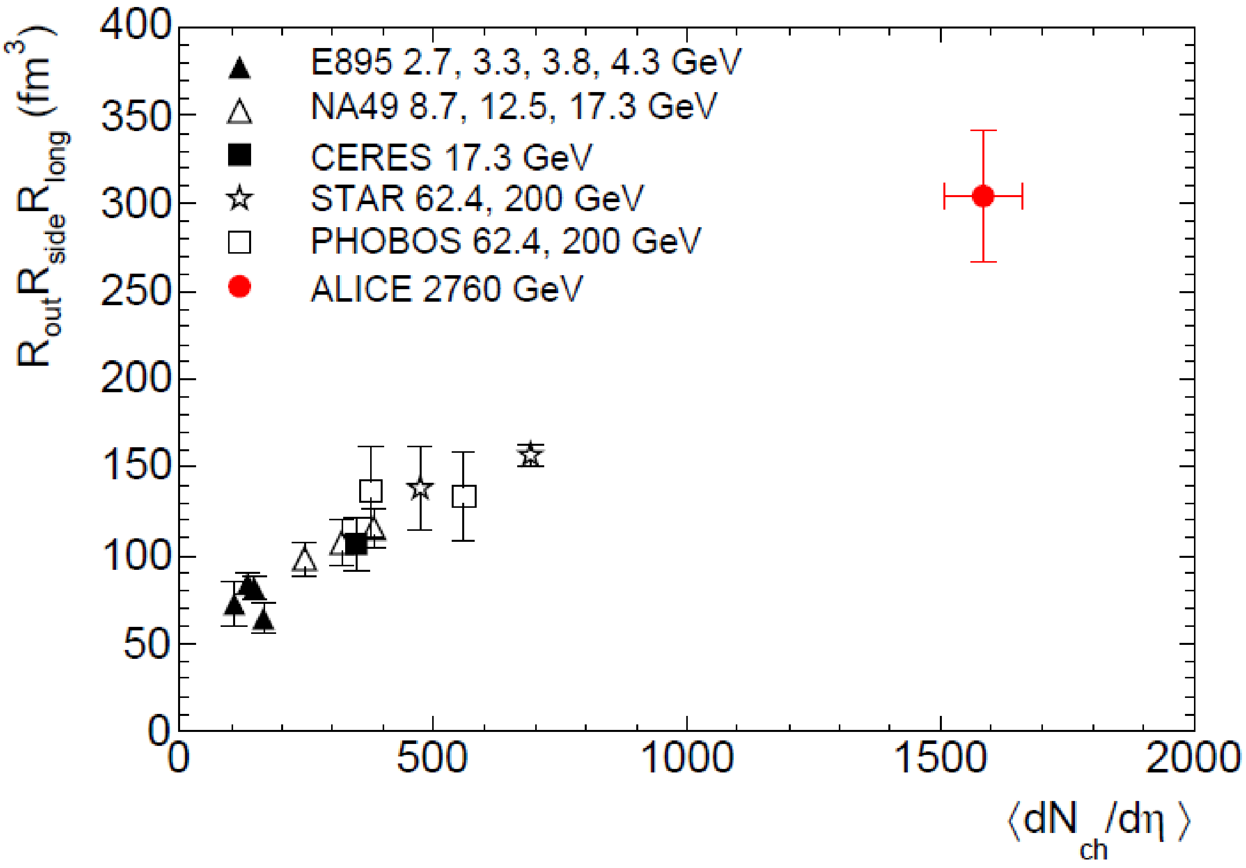
for reference:

$R \simeq 1$ fm in pp collisions

from A. Andronic

Volume in central collisions

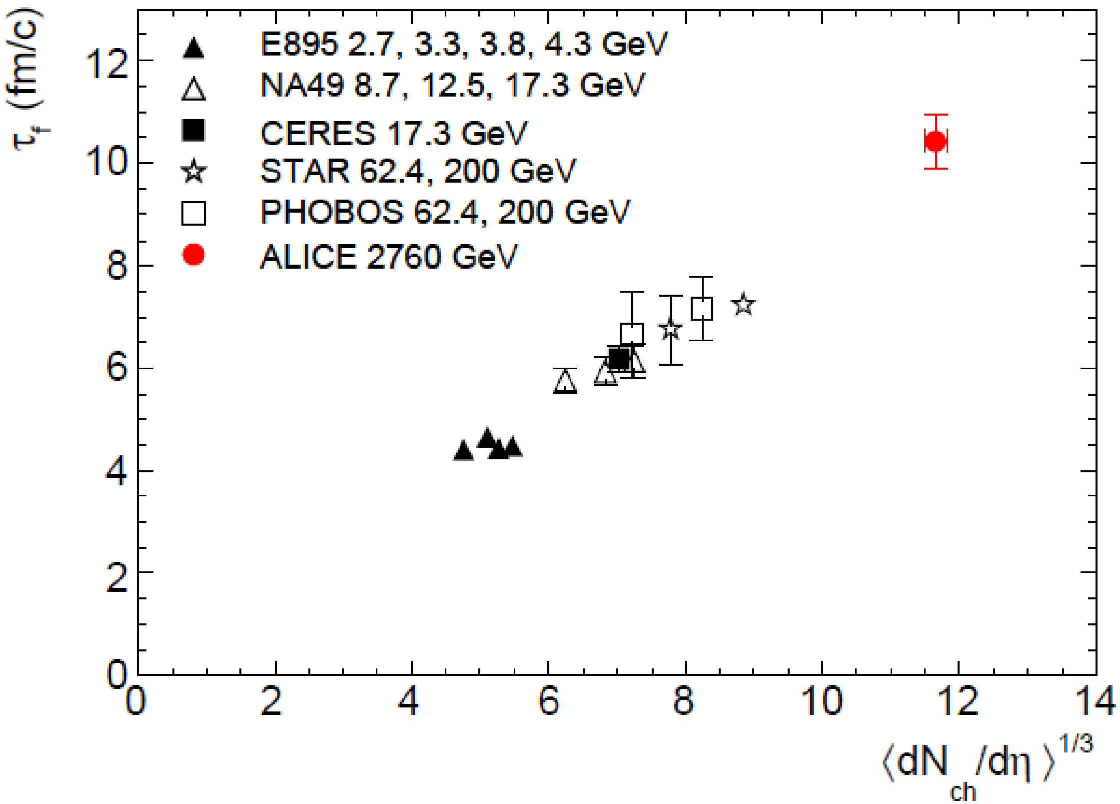
(∼ volume of the homogeneity region)



ALICE collab., arXiv:1012.4035

from A. Andronic

Decoupling time



ALICE collab., arXiv:1012.4035

Fit R_{long} with:

$$R_{\text{long}}{}^2(k_T) = \frac{\tau_f^2 T}{m_T} \frac{K_2(m_T/T)}{K_1(m_T/T)}$$

$$m_T = \sqrt{m_\pi^2 + k_T^2}$$

T the kinetic freeze-out temperature (120 MeV),

K_1 and K_2 mod. Bessel f.

The size of the homogeneity region is inversely proportional to the velocity gradient of the expanding system. The longitudinal velocity gradient in a high energy nuclear collision decreases with time as $1/\tau$. R_{long} is proportional to the total duration of the longitudinal expansion, τ_f.

from A. Andronic